A Stewart Platform- Based Manipulator: General Theory and Practical Construction

E.F. Fichter

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E. F. Fichter
Industrial Engineering Department
Oregon State University
Corvallis, Oregon 97331

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Abstract

The Stewart Platform is one example of a parallel connection robot manipulator. This paper summarizes work that has been done at Oregon State University over the past several years on this topic. The work has fallen into two general areas: theoretical consideration of the generalized Stewart Platform and practical considerations for building a working machine. Both of these areas are covered in this paper.

The theoretical part of the paper discusses the following four problems:

1. *Given the position and orientation of the end effector, what are the actuator coordinates?*
2. *Given the velocity, position, and orientation of the end effector, what are the actuator velocities?*
3. *Given the forces exerted on the end effector by the external world and the accelerations of the end effector, what are the forces at the actuators?*
4. *What are the singular configurations of the manipulator? For the Stewart Platform, the singular configurations are positions where the end effector gains one or more degrees of freedom.*

After the theoretical problems have been solved, there are still a group of practical problems to be tackled when a real machine is built. How can the general configuration be simplified to make the solution of the equations practical? Once a configuration is decided upon, what are the construction considerations for building a real machine? What is the range of motion of the end effector? What are the necessary ranges of motion of the joints? What are the implications of the singularities of the Stewart Platform and what are some practical ways of solving the problems they cause?

In the process of doing this research, three machines were built and a computer simulation was written and used. The most recent machine and the computer simulation will be described in this paper.

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1. Introduction

Most of the literature on robot manipulators is concerned with serially connected mechanisms, that is, mechanisms whose links and joints alternate with one another in a long chain. Parallel connection is an alternative type, where the links and joints form two or more serially connected chains; these chains connect the base of the manipulator with the end effector. Figure 1 illustrates simple examples of these two configurations. A start at classifying the many possible parallel connection manipulators has been presented by Earl and Rooney (1983). There are also combinations in which part of the manipulator is serial and part parallel (Hunt 1983).

The subject of this paper is a parallel connection manipulator called the *Stewart Platform,* illustrated simply in Fig. 2. The basic reference for this mechanism is the paper by Stewart (1965), which suggests using it as an aircraft simulator motion base. Dimentberg (1965) uses the body supported by six rods as an example application of screw calculus. Hunt (1978) suggests its use as a manipulator and mentions some of its advantages and disadvantages. Fichter and McDowell (1980) describes some of its advantages and disadvantages as a manipulator in greater detail. Fichter and McDowell (1983) describes a technique for determining the value of the joint variable for every joint of the mechanism, not only the powered joints. This information is valuable for the practical design of the joints in each of the supporting legs. Hunt (1983) discusses some alternative mechanical designs for the Stewart Platform. A group in Great Britain designed and built a manipulator based on the Stewart Platform

Fig. 1. Simple planar (2-dimensional) examples of series- and parallel-connected manipulators.

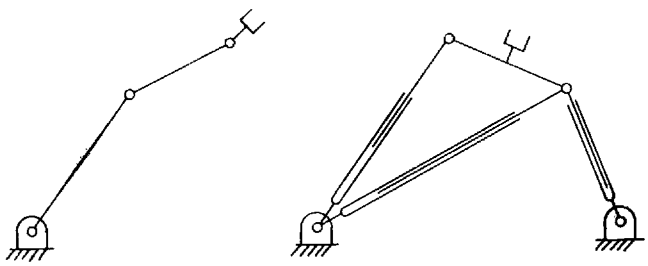
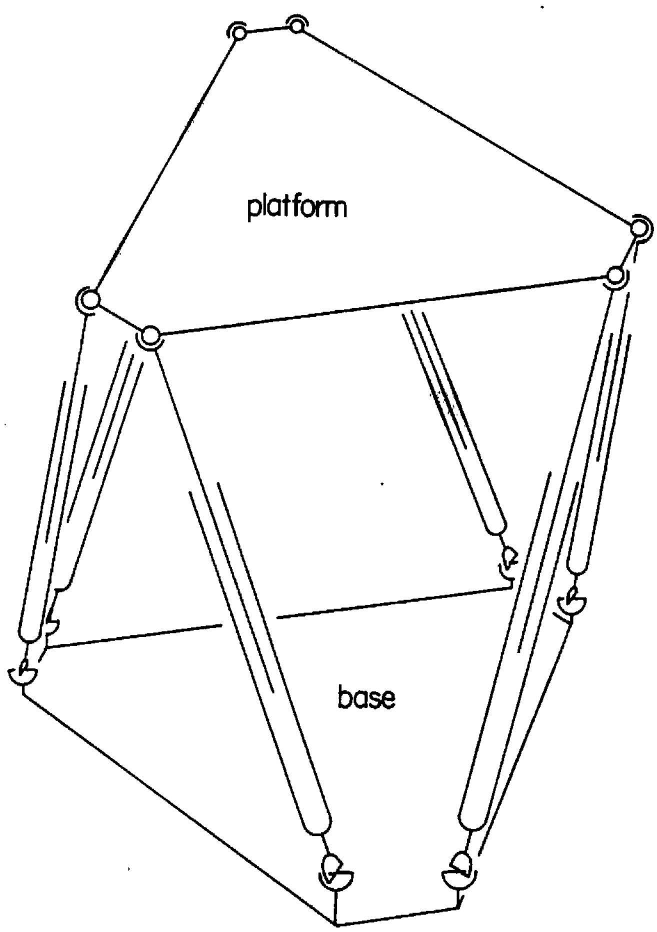


Fig. 2. One of the common kinematic arrangements of the Stewart Platform.



(Powell 1981, Potton 1983). Yang and Lee (1984) presents a solution to the kinematics of a simplified version of the Stewart Platform but does not consider the general Stewart Platform. Yang and Lee (1984) also presents some preliminary information on a simulation. In a previous paper Fichter (1984) presented some of the material on kinematics and differential kinematics presented here, but this material has been clarified and expanded for this paper.

The first part of the paper addresses the following four questions:

1. Given a position and orientation of the end effector, what are the necessary actuator coordinates? This is sometimes called the *inverse kinematics* problem for serially connected manipulators.
2. Given a position, orientation, and velocity of the end effector, what are the necessary actuator velocities?
3. Given a position, orientation, and external force and torque on the end effector, what are the necessary actuator forces?
4. Is the manipulator singular for a particular position and orientation of the end effector?

These first four questions are initially considered for the completely general Stewart Platform.

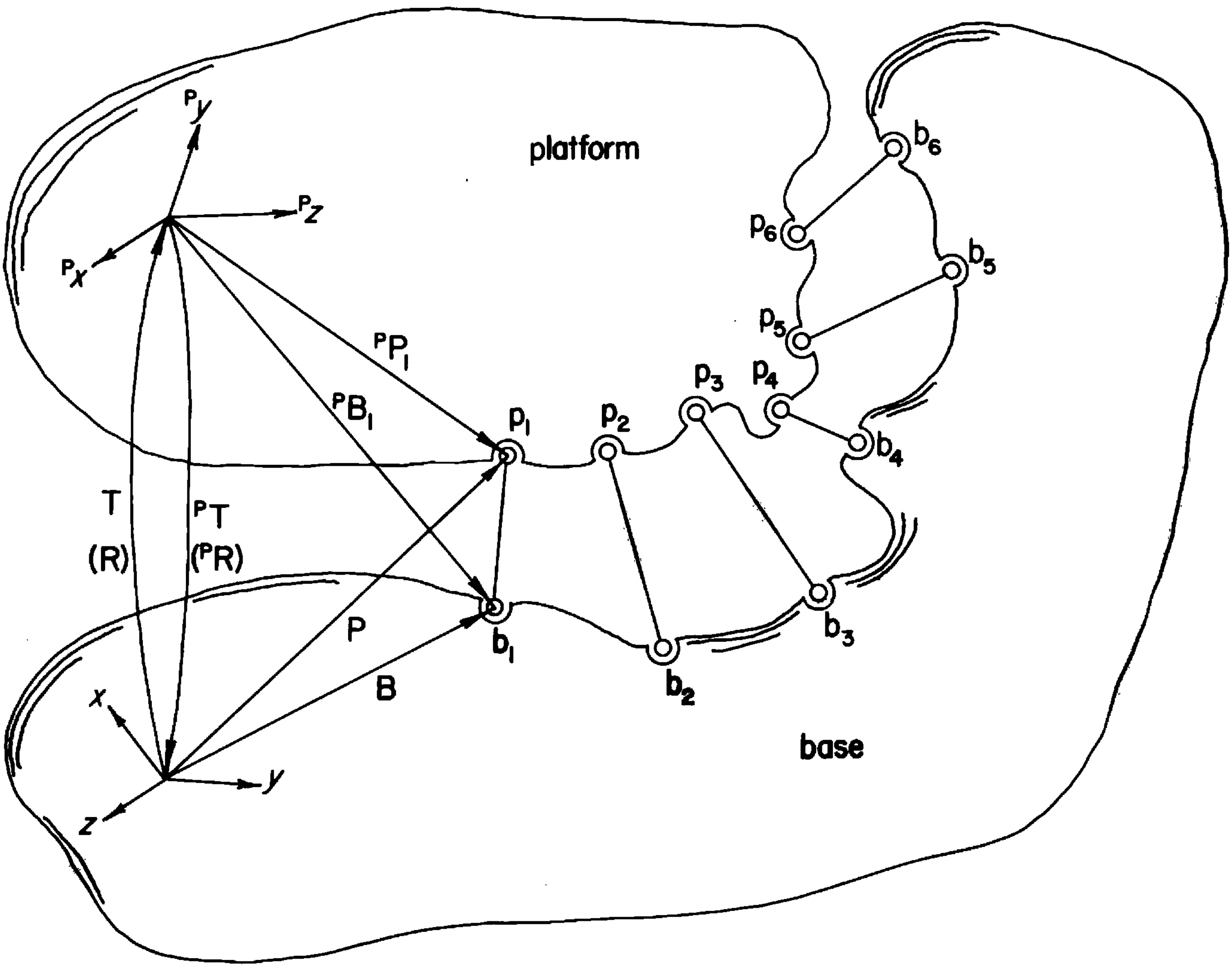
In the last part of this paper, the problem considered is how to build a practical manipulator. This discussion includes the simplification of the general equations derived in answering the first four questions, a description of a computer simulation of the Stewart Platform, and a description of a prototype machine.

2. Kinematic Equations for the General Stewart Platform

The Stewart Platform consists of two bodies connected by six legs, which can vary in length (Fig. 3). One of the bodies is called the *base* and the other is called the *platform*. Notice the topological symmetry of each body relative to the other. This symmetry makes the assignment of these names arbitrary.

Each of the six legs has one of its end points fixed in the base and the other end point fixed in the platform. The locations of these six points in each body are chosen arbitrarily. (There are a few restrictions on the choice of positions. For instance, if the points in the base and in the platform are at the corners of regular planar hexagons, the platform is not always con-

Fig. 3. A generalized Stewart Platform with the base and platform coordinate systems shown. The six points in the base and the six points in the platform at the ends of the legs are illustrated along with the vector descriptions of points $\mathbf{b}_1$ *and* $\mathbf{p}_1$ *in both coordinate systems. The relationships between the coordinate systems* $\mathbf{T}$, $\mathbf{R}$, ${}^{P}\mathbf{T}$, *and* ${}^{P}\mathbf{R}$ *are also illustrated.*



strained relative to the base.) A right-handed coordinate system is defined at a convenient place in each body (Fig. 3).

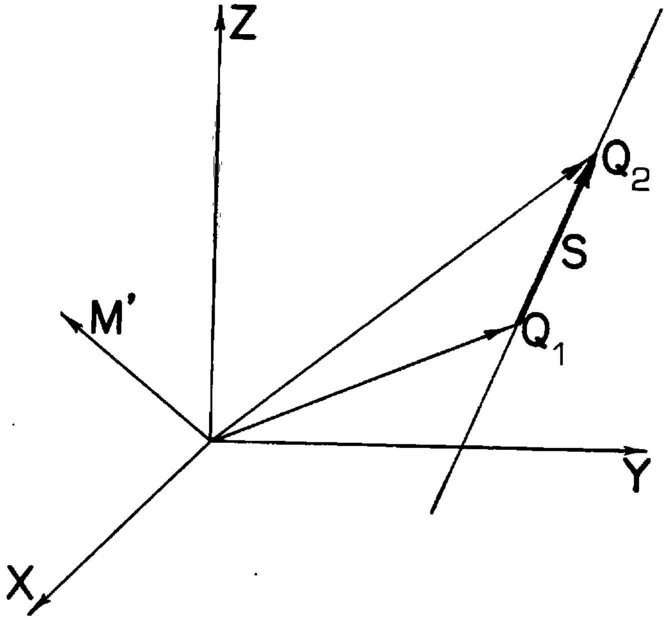
Each of the six points in the base is described by a position vector, $\mathbf{B}_i$, referenced to the base coordinate system. (Definitions for all symbols will be found in the Appendix.) Similarly, each of the points in the platform is described by a position vector ${}^{P}\mathbf{P}_i$, referenced to the platform coordinate system. The left superscript will be used to indicate the coordinate system to which a particular vector is referenced; $\mathbf{P}$ is for platform, and no left superscript is for base. Vectors will be treated as row matrices unless otherwise noted.

The orientation of the platform relative to the base is defined by a rotation matrix, $\mathbf{R}$.

$$\mathbf{R} = \begin{bmatrix} \alpha_x & \beta_x & \gamma_x \\ \alpha_y & \beta_y & \gamma_y \\ \alpha_z & \beta_z & \gamma_z \end{bmatrix} \tag{1}$$

The column vectors, α, β, and γ, that make up this matrix are the unit vectors along the x-axis, y-axis, and z-axis, respectively, of the platform coordinate system. The matrix $\mathbf{R}$ can also be used to transform free vectors from the base coordinate system to the platform

Fig. 4. The unnormalized Plücker coordinates of the line through points $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *can be decomposed into two three-component vectors,* **S** *and* $\mathbf{M}'$.



coordinate system. The matrix **R** will be used in both of these ways: as a description of the orientation of the platform relative to the base and as a transformation matrix.

The position of the platform relative to the base is defined by a translation that may be written as a vector, **T**, from the origin of the base coordinate system to the origin of the platform coordinate system:

$$\mathbf{T} = [T_x \quad T_y \quad T_z]. \tag{2}$$

Similarly, the orientation of the base relative to the platform is defined by a rotation matrix, ${}^p\mathbf{R}$:

$${}^p\mathbf{R} = \mathbf{R}^{-1} = \mathbf{R}^T = \begin{bmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{bmatrix}. \tag{3}$$

The column vectors making up this matrix are the unit vectors along the x-axis, y-axis, and z-axis, respectively, of the base coordinate system. The matrix ${}^p\mathbf{R}$ will be used in two ways: as a description of the orientation of the base relative to the platform and as a transformation matrix.

The position of the platform relative to the base is defined by a translation that may be written as a vector, ${}^p\mathbf{T}$, from the origin of the platform coordinate system to the origin of the base coordinate system:

$${}^p\mathbf{T} = -\mathbf{T}\,\mathbf{R}. \tag{4}$$

In this equation, the rotation transformation **R** is used to find the components of translation vector **T** in the platform coordinate system. The minus sign is required since the vector ${}^p\mathbf{T}$ is in the opposite direction from the vector **T**.

Equations (1) and (2) give a complete description of the platform coordinate system referenced to the base coordinate system, while Eqs. (3) and (4) give a complete description of the base coordinate system referenced to the platform coordinate system.

It is convenient to treat the legs as lines and to represent them as line coordinates. The most applicable set of line coordinates are the Plücker coordinates, which may be determined from any two distinct points, $\mathbf{Q}_1$ and $\mathbf{Q}_2$, on the line as shown in Fig. 4 and in the equations below:

$$\mathbf{S} = \mathbf{Q}_2 - \mathbf{Q}_1, \tag{5}$$

$$\mathbf{M}' = \mathbf{Q}_1 \times \mathbf{Q}_2 = \mathbf{Q}_1 \times \mathbf{S} = \mathbf{Q}_2 \times \mathbf{S}. \tag{6}$$

The vector **S** lies along the line and the vector $\mathbf{M}'$ is perpendicular to the plane containing the line and the origin. The vector $\mathbf{M}'$ is the moment of the vector **S** about the origin. These two three-component vectors are assembled into the six-component vector of Plücker coordinates of the line:

$$\mathbf{U}' = [S_x \quad S_y \quad S_z \quad M'_x \quad M'_y \quad M'_z]. \tag{7}$$

It will be useful to normalize the Plücker coordinates with respect to the magnitude of the vector **S**:

$$\mathbf{s} = \frac{\mathbf{S}}{|\mathbf{S}|}, \tag{8}$$

$$\mathbf{M} = \frac{\mathbf{M}'}{|\mathbf{S}|}, \tag{9}$$

where

$$|\mathbf{S}| = \sqrt{\mathbf{S} \cdot \mathbf{S}}.$$

The vector $\mathbf{s}$ is the unit vector in the direction of the line, and the magnitude of the vector $\mathbf{M}$ is the shortest distance from the origin to the line. Thus vector $\mathbf{M}$ is the moment about the origin of a unit force acting along the line. The vectors $\mathbf{s}$ and $\mathbf{M}$ can be assembled into the normalized Plücker coordinate vector:

$$\mathbf{U} = [s_x \quad s_y \quad s_z \quad M_x \quad M_y \quad M_z]. \tag{10}$$

An alternative presentation of the Plücker coordinates is given by Hunt (1978).

2.1. Determining Leg Lengths, Directions, and Moments

The lengths, directions, and moments of the legs can be referenced to either the base coordinate system or the platform coordinate system. First, equations for these quantities referenced to the base coordinate system will be derived and then similar equations for the same quantities referenced to the platform coordinate systems will be derived.

To determine these three quantities, we begin by writing an equation for the vector, $\mathbf{S}_i$, from $\mathbf{B}_i$ to $\mathbf{P}_i$:

$$\mathbf{S}_i = \mathbf{P}_i - \mathbf{B}_i.$$

In this equation, the point in the platform is referenced to the base coordinate system. Hence, the coordinates of the platform point must be transformed to the base coordinate system:

$$\mathbf{P}_i = \mathbf{T} + ({}^p\mathbf{P}_i\, {}^p\mathbf{R}),$$
$$\mathbf{P}_i = \mathbf{T} + ({}^p\mathbf{P}_i\, \mathbf{R}^T).$$

The first term on the right side of this equation accounts for the translation from the origin of the base coordinate system to the origin of the platform coordinate system. The second term on the right side of this equation transforms the components of the vector ${}^p\mathbf{P}_i$ from the platform coordinate system to the base coordinate system. These equations may be combined into the equation below for the leg vector, $\mathbf{S}_i$:

$$\mathbf{S}_i = \mathbf{T} + ({}^p\mathbf{P}_i\, \mathbf{R}^T) - \mathbf{B}_i. \tag{11}$$

The length of the leg, σ_i, is the magnitude of vector $\mathbf{S}_i$:

$$\sigma_i = |\mathbf{S}_i| = \sqrt{\mathbf{S}_i \cdot \mathbf{S}_i}. \tag{12}$$

The direction of the leg referenced to the base coordinate system, $\mathbf{s}_i$, is the unit vector along $\mathbf{S}_i$:

$$\mathbf{s}_i = \frac{\mathbf{S}_i}{\sigma_i}. \tag{13}$$

The normalized moment of the leg vector about the origin of the base coordinate system, $\mathbf{M}_i$, can be found by substituting into Eq. (6):

$$\mathbf{M}_i = \mathbf{B}_i \times \mathbf{s}_i = [\mathbf{T} + ({}^p\mathbf{P}_i\, \mathbf{R}^T)] \times \mathbf{s}_i. \tag{14}$$

In this equation, the unit vector, $\mathbf{s}_i$, is used so that the result is the normalized moment vector as defined in Eq. (9). The above equation gives the normalized moment vector referenced to the base coordinate system in terms of either end point of the leg.

This completes the determination of the leg lengths, directions, and moments referenced to the base coordinate system. Before we go on to the determination of these quantities referenced to the platform coordinate system, it should be noted that the direction, $\mathbf{s}_i$, from Eq. (13) and the normalized moment, $\mathbf{M}_i$, from Eq. (14) can be combined using Eq. (10) to form the normalized Plücker coordinates, $\mathbf{U}_i$, of the leg referenced to the base coordinate system.

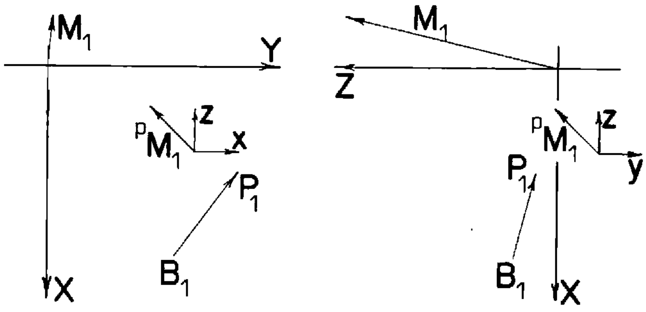
The equation for the leg vector referenced to the platform coordinate system can be written using a derivation that is parallel to the derivation used for the leg vector equation referenced to the base coordinate system:

$${}^p\mathbf{S}_i = {}^p\mathbf{P}_i - {}^p\mathbf{B}_i,$$
$${}^p\mathbf{B}_i = {}^p\mathbf{T} + (\mathbf{B}_i\, \mathbf{R}).$$

If we eliminate ${}^p\mathbf{B}_i$,

$${}^p\mathbf{S}_i = {}^p\mathbf{P}_i - {}^p\mathbf{T} - (\mathbf{B}_i\, \mathbf{R}). \tag{15}$$

Fig. 5. The front and right side view of the configuration discussed in the first example. The coordinate system in the base is labeled with uppercase letters and the coordinate system in the platform is labeled with lowercase letters.



Alternatively, since $\mathbf{S}_i$ is a line vector, it may be transformed to the platform coordinate system using the orientation relationship $\mathbf{R}$.

$$^p\mathbf{S}_i = \mathbf{S}_i\,\mathbf{R}. \tag{16}$$

The equivalence of Eqs. (15) and (16) can be verified as follows: substitute for $^p\mathbf{T}$ in Eq. (15) using Eq. (4) and collect terms:

$$^p\mathbf{S}_i = {}^p\mathbf{P}_i + [(\mathbf{T} - \mathbf{B}_i)\,\mathbf{R}].$$

Use the identity $\mathbf{R}^T\,\mathbf{R}$, factor out $\mathbf{R}$, and rearrange:

$$^p\mathbf{S}_i = [\mathbf{T} + ({}^p\mathbf{P}_i\,\mathbf{R}^T) - \mathbf{B}_i]\mathbf{R}.$$

From Eq. (11), the factor in brackets is equal to $\mathbf{S}_i$. This verifies the equivalence of Eqs. (15) and (16).

Since the leg vector is the same vector no matter what coordinate system it is referenced to, the length of the leg, σ_i, in either coordinate system must be the same:

$$\sigma_i = |{}^p\mathbf{S}_i| = \sqrt{{}^p\mathbf{S}_i \cdot {}^p\mathbf{S}_i} = \sqrt{{}^p\mathbf{S}_i\,{}^p\mathbf{S}_i^T}. \tag{17}$$

The identity of σ_i in Eqs. (12) and (17) can be verified by substituting Eq. (16) into the far right side of Eq. (17).

The direction of the leg referenced to the platform coordinate system, $^p\mathbf{s}_i$, is the unit vector along $^p\mathbf{S}_i$:

$$^p\mathbf{s}_i = \frac{^p\mathbf{S}_i}{\sigma_i} = \mathbf{s}_i\,\mathbf{R}. \tag{18}$$

The normalized moment of the leg vector about the origin of the platform coordinate system, $^p\mathbf{M}_i$, can be found by substituting into Eq. (6):

$$^p\mathbf{M}_i = {}^p\mathbf{P}_i \times {}^p\mathbf{s}_i = [{}^p\mathbf{T} + (\mathbf{B}_i\,\mathbf{R})] \times {}^p\mathbf{s}_i. \tag{19}$$

This equation is similar to Eq. (14) except that it gives the normalized moment vector of the leg vector about the origin of the platform coordinate system with the moment vector referenced to the platform coordinate system.

As above, the normalized Plücker coordinates, $^p\mathbf{U}_i$, of the leg are obtained by combining $^p\mathbf{s}_i$ from Eq. (18) and $^p\mathbf{M}_i$ from Eq. (19). These Plücker coordinates are referenced to the platform coordinate system.

A relatively simple relationship exists between the two vectors $\mathbf{M}_i$ and $^p\mathbf{M}_i$. This relationship is derived here beginning with Eq. (19):

$$^p\mathbf{M}_i = [{}^p\mathbf{T} + (\mathbf{B}_i\,\mathbf{R})] \times {}^p\mathbf{s}_i.$$

Expand with Eqs. (4) and (18) and collect terms:

$$\begin{aligned} ^p\mathbf{M}_i &= [(-\mathbf{T} + \mathbf{B}_i)\,\mathbf{R}] \times (\mathbf{s}_i\,\mathbf{R}), \\ &= [(\mathbf{B}_i - \mathbf{T}) \times \mathbf{s}_i]\,\mathbf{R}, \\ &= [(\mathbf{B}_i \times \mathbf{s}_i) - (\mathbf{T} \times \mathbf{s}_i)]\,\mathbf{R}. \end{aligned}$$

Simplify this using Eq. (14) to obtain

$$^p\mathbf{M}_i = [\mathbf{M}_i - (\mathbf{T} \times \mathbf{s}_i)]\,\mathbf{R}. \tag{20}$$

This completes the derivation of the equations for leg length, direction, and moment. Equations (12) and (17) with Eq. (11) and either Eqs. (15) or (16) answer the first question cited in the introduction; given the position and orientation of the platform, what are the necessary actuator coordinates?

An example of how these equations are used follows (see Fig. 5). The coordinates of the end points of the one leg to be considered in this example are given below:

$$\begin{aligned} \mathbf{B}_1 &= [9 \quad 6 \quad 2], \\ ^p\mathbf{P}_1 &= [2 \quad -3 \quad -1]. \end{aligned}$$

The rotational and translational relationships between the two bodies are given below. The equations in this example are labeled with the general equation numbers primed.

$$\mathbf{R}=\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad (1')$$

$$\mathbf{T}=[\ 4 \quad 7 \quad -2], \qquad (2')$$

$$^{p}\mathbf{R}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \qquad (3')$$

$$^{p}\mathbf{T}=-[\ 4 \quad 7 \quad -2]\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad (4')$$

$$=[-7 \quad -2 \quad 4].$$

The vector along the leg in the base coordinate system and in the platform coordinate system is determined as follows:

$$\mathbf{S}_1=[2 \quad -3 \quad -1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} + [4 \quad 7 \quad -2]-[9 \quad 6 \quad 2], \qquad (11')$$

$$\mathbf{S}_1=[-4 \quad 3 \quad -1],$$

$$^{p}\mathbf{S}_1=[2 \quad -3 \quad -1]+([4 \quad 7 \quad -2] - [9 \quad 6 \quad 2])\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad (15')$$

$$^{p}\mathbf{S}_1=[3 \quad 1 \quad 4],$$

$$^{p}\mathbf{S}_1=[-4 \quad 3 \quad -1]\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$=[3 \quad 1 \quad 4]. \qquad (16')$$

The last equation is a verification of the results of Eq. (15′). The length of the leg can be found by using either Eq. (12) or Eq. (17).

$$\sigma_1=\sqrt{(-4)^2+3^2+(-1)^2}=5.1 \qquad (12')$$

$$\sigma_1=\sqrt{3^2+1^2+4^2}=5.1 \qquad (17')$$

The unit vectors in the base and the platform coordinate systems are determined from Eqs. (13) and (18):

$$\mathbf{s}_1=\frac{1}{5.1}[-4 \quad 3 \quad -1]=[-0.78 \quad 0.59 \quad -0.20], \qquad (13')$$

$$^{p}\mathbf{s}_1=\frac{1}{5.1}[3 \quad 1 \quad 4]=[0.59 \quad 0.20 \quad 0.78]. \qquad (18')$$

The moment of the leg about the origin of the base coordinate system may be calculated using either of the two parts of Eq. (14):

$$\mathbf{M}_1=[9 \quad 6 \quad 2]\times[-0.78 \quad 0.59 \quad -0.20], \qquad (14A')$$

$$\mathbf{M}_1=[-2.38 \quad 0.24 \quad 9.99],$$

$$\mathbf{M}_1=\left\{[2 \quad -3 \quad -1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} + [4 \quad 7 \quad -2]\right\}\times[-0.78 \quad 0.59 \quad -0.20] \qquad (14B')$$

$$=[5 \quad 9 \quad 1]\times[-0.78 \quad 0.59 \quad -0.20],$$

$$\mathbf{M}_1=[-2.39 \quad 0.22 \quad 9.97].$$

The moment vectors calculated in the two parts of Eq. (14′) differ slightly because of round-off error. The moment of the leg about the origin of the platform coordinate system may be calculated using either of the two parts of Eq. (19) (only the first part of Eq. (19) will be presented in this example).

$$^{p}\mathbf{M}_1=[2 \quad -3 \quad -1]\times[0.59 \quad 0.20 \quad 0.78], \qquad (19A')$$

$$^{p}\mathbf{M}_1=[-2.14 \quad -2.15 \quad 2.17].$$

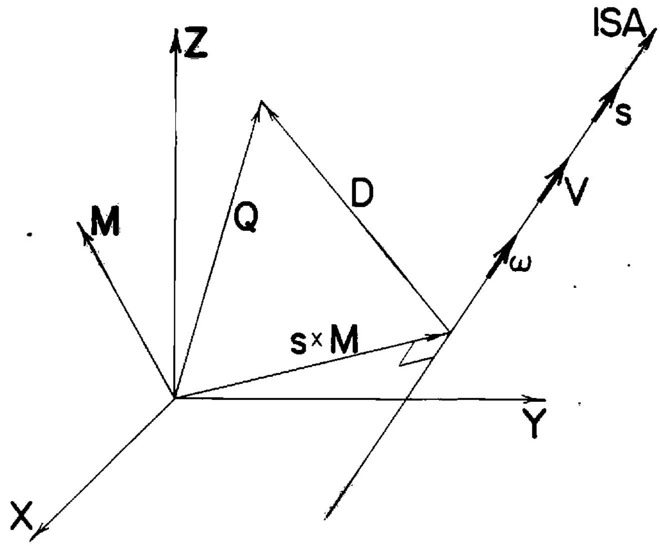
This result can be verified by using Eq. (20):

$$^{p}\mathbf{M}_1=\{[-2.38 \quad 0.24 \quad 9.99]-([4 \quad 7 \quad -2] \times[-0.78 \quad 0.59 \quad -0.20])\}\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad (20')$$

$$^{p}\mathbf{M}_1=[-2.12 \quad -2.17 \quad 2.16].$$

This completes the examples of the use of these equations.

Fig. 6. The ISA *of a body is shown with the velocity,* **V**, *along the* ISA*; the angular velocity,* $\boldsymbol{\omega}$, *about the* ISA*; the unit vector,* **s**, *along the* ISA*; and the moment,* **M**, *of the* ISA *about the origin. A point,* **Q**, *in the body defined by vector* **Q** *is related to the* ISA *by the vector* **D**.



2.2. Determining the Components of Leg Velocity

The components of the velocity of a leg are the rate of change of length of the leg vector and the rate of change of the direction of the vector. These are components of the vector rate of change of the leg vector. The vector rate of change will be determined using screw theory. It can be verified that any motion of a body in three-dimensional space is a screwing motion, as described by Hunt (1978). At every instant during the motion of a body in space, there is an instantaneous screw axis (*ISA*), a unique line that the body is rotating about and translating along. In the general case, when the body is both rotating and translating, the angular velocity vector for the body and the linear velocity vector for a point in the body are parallel if, and only if, the point lies on the *ISA*. The pitch, h, of the *ISA* is the ratio of the linear velocity, **V**, of any point on the *ISA* to the angular velocity of the body, $\boldsymbol{\omega}$.

$$\mathbf{V} = h\boldsymbol{\omega}. \tag{21}$$

If there is no linear velocity, the pitch is zero; and if there is no angular velocity, the pitch is infinite.

If the Plücker coordinates, or vectors **s** and **M**, of the *ISA* are known along with any two of the three quantities h, **V**, and $\boldsymbol{\omega}$, then the velocity of any point in the moving body can be found by using the following equation. (Vectors **s** and **M** referred to here are not related to any of the legs, but describe the *ISA* of the platform.)

$$\dot{\mathbf{Q}} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{D}.$$

These vectors are shown in Fig. 6. The vector **D** originates at the head of the vector labeled $\mathbf{s} \times \mathbf{M}$ and terminates at point **Q**. Vector **D** is not necessarily perpendicular to the *ISA*; it is eliminated below:

$$\dot{\mathbf{Q}} = \mathbf{V} + \boldsymbol{\omega} \times [\mathbf{Q} - (\mathbf{s} \times \mathbf{M})]. \tag{22}$$

This equation is used to find the velocity of point ${}^{p}\mathbf{P}_i$ referenced to the base coordinate system by replacing vector **Q** with point $\mathbf{P}_i$, as in Eq. (11).

$$\dot{\mathbf{P}}_i = \mathbf{V} + \boldsymbol{\omega} \times [\mathbf{T} + ({}^{p}\mathbf{P}_i\mathbf{R}^T) - (\mathbf{s} \times \mathbf{M})].$$

Since the other end of the leg is fixed relative to the base coordinate system, this is the equation for the vector velocity of the leg vector referenced to the base coordinate system.

$$\dot{\mathbf{S}}_i = \mathbf{V} + \boldsymbol{\omega} \times [\mathbf{T} + ({}^{p}\mathbf{P}_i\mathbf{R}^T) - (\mathbf{s} \times \mathbf{M})]. \tag{23}$$

Now this vector leg velocity can be broken into its components. Figure 7 shows the relationship between the leg vector, **S**, and the leg velocity vector, $\dot{\mathbf{S}}$; it also shows the components of the leg velocity vector, which are given algebraically below:

$$\dot{\sigma}_i = \dot{\mathbf{S}}_i \cdot \mathbf{s}_i. \tag{24}$$

The equation gives the magnitude of the rate of change of length of the leg. To determine the vector rate of change of direction of the leg, $\dot{\mathbf{s}}_i$, the vector rate of change of the length of the leg is required; this is just $\dot{\sigma}_i\mathbf{s}_i$.

$$\dot{\mathbf{s}}_i = \dot{\mathbf{S}}_i - \dot{\sigma}_i\mathbf{s}_i. \tag{25}$$

Note that $\dot{\mathbf{s}}_i$ is not a unit vector. The angular velocity of the leg, $\dot{\mathbf{A}}_i$, has the magnitude equal to the length of $\dot{\mathbf{s}}_i$ divided by the length of the leg and the direction of the cross-product of $\mathbf{s}_i$ and $\dot{\mathbf{s}}_i$. It is given by the following equation:

$$\dot{\mathbf{A}}_i = \frac{1}{\sigma_i}(\mathbf{s}_i \times \dot{\mathbf{s}}_i). \tag{26}$$

The following equations give the same results as Eqs. (23)–(26), but with reference to the platform coordinate system.

$${}^p\dot{\mathbf{S}}_i = {}^p\mathbf{V} + {}^p\omega \times [{}^p\mathbf{P}_i - ({}^p\mathbf{s} \times {}^p\mathbf{M})]. \tag{27}$$

The leg velocity vector referenced to the platform coordinate system can also be written directly in terms of the leg velocity vector referenced to the base coordinate system and the rotation transformation between the two coordinate systems.

$${}^p\dot{\mathbf{S}}_i = \dot{\mathbf{S}}_i\mathbf{R}. \tag{28}$$

The verification of the identity of Eqs. (27) and (28) follows. Begin by transforming the free vectors in Eq. (27).

$${}^p\dot{\mathbf{S}}_i = (\mathbf{VR}) + (\omega\mathbf{R}) \times [{}^p\mathbf{P}_i - \{(\mathbf{sR}) \times [(\mathbf{M} - [\mathbf{T} \times \mathbf{s}])\mathbf{R}]\}]$$

The second term in the curly braces comes from Eq. (20). The terms in the braces can be simplified as follows.

$$\begin{aligned}
&(\mathbf{sR}) \times [(\mathbf{M} - [\mathbf{T} \times \mathbf{s}])\mathbf{R}] \\
&\{\mathbf{s} \times (\mathbf{M} - [\mathbf{T} \times \mathbf{s}])\}\mathbf{R} \\
&\{(\mathbf{s} \times \mathbf{M}) - (\mathbf{s} \times [\mathbf{T} \times \mathbf{s}])\}\mathbf{R} \\
&\{(\mathbf{s} \times \mathbf{M}) - [(\mathbf{s} \cdot \mathbf{s})\mathbf{T}] + [(\mathbf{s} \cdot \mathbf{T})\mathbf{s}]\}\mathbf{R}
\end{aligned}$$

Since $\mathbf{s}$ is a unit vector, the second term inside the braces in the factor above is $-\mathbf{T}$. Substitute this factor into the equation above and factor out $\mathbf{R}$.

$${}^p\dot{\mathbf{S}}_i = \{\mathbf{V} + \omega \times [\mathbf{T} + ({}^p\mathbf{P}_i\mathbf{R}^T) - (\mathbf{s} \times \mathbf{M}) - \{(\mathbf{s} \cdot \mathbf{T})\mathbf{s}\}]\}\mathbf{R}$$

The last term in the square brackets will disappear since it is parallel to the angular velocity, ω, (Fig. 6), leaving the factor in the braces equal to $\dot{\mathbf{S}}_i$ from Eq. (23).

The magnitude of the rate of change of leg length referenced to the platform coordinate system is given below:

$$\dot{\sigma}_i = {}^p\dot{\mathbf{S}}_i \cdot {}^p\mathbf{s}_i = {}^p\dot{\mathbf{S}}_i{}^p\mathbf{s}_i^T. \tag{29}$$

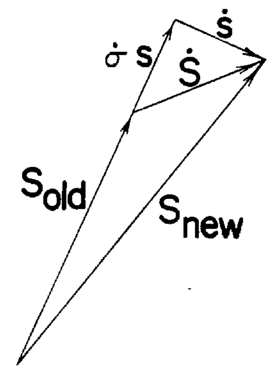


Fig. 7. The vector rate of change, $\dot{\mathbf{S}}$, of vector $\mathbf{S}$ is shown along with the components of $\dot{\mathbf{S}}$ in the direction of $\mathbf{S}$ and perpendicular to $\mathbf{S}$.

By substituting Eqs. (28) and (18) into the far right side of Eq. (29), it can be verified that the rate of change of the leg length referenced to the platform coordinate system is the same as the rate of change of leg length referenced to the base coordinate system. The vector rate of change of direction, ${}^p\dot{\mathbf{s}}_i$, and the angular velocity, ${}^p\dot{\mathbf{A}}_i$, of the leg referenced to the platform coordinate system can either be found in a manner parallel to Eqs. (25) and (26) or by transformation of these quantities referenced to the base coordinate system. Both definitions are given below:

$${}^p\dot{\mathbf{s}}_i = {}^p\dot{\mathbf{S}}_i - \dot{\sigma}_i{}^p\mathbf{s}_i = \dot{\mathbf{s}}_i\mathbf{R}, \tag{30}$$

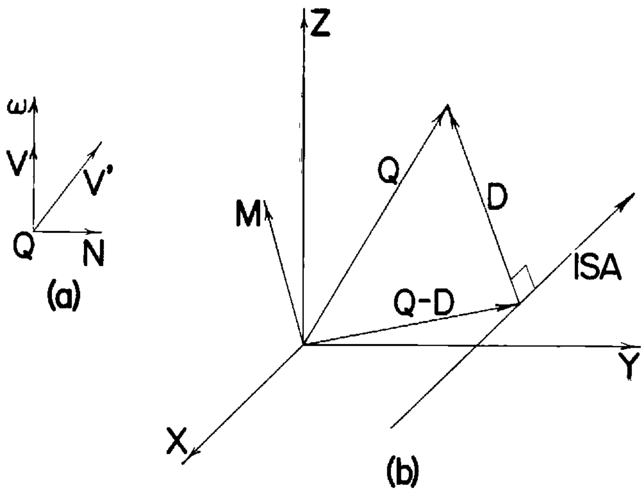
$${}^p\dot{\mathbf{A}}_i = \frac{1}{\sigma_i}({}^p\mathbf{s}_i \times {}^p\dot{\mathbf{s}}_i) = \dot{\mathbf{A}}_i\mathbf{R}. \tag{31}$$

The above equations give the components of the leg velocity in terms of the *ISA*, the linear velocity, $\mathbf{V}$, of the body in the direction of the *ISA*, and the angular velocity, ω, of the body. If these parameters are not known, then the components of the leg velocity can still be determined if the angular velocity, ω, of the body and the linear velocity, $\mathbf{V}'$, of some point, $\mathbf{Q}$, in the body are known.

Consider a point, $\mathbf{Q}$, not on the *ISA*; its linear velocity is composed of two components, one parallel to the *ISA* and one perpendicular to the *ISA*, as shown in Fig. 8A. The first component is the linear velocity of the body along the *ISA*, $\mathbf{V}$, given by the following equation:

$$\mathbf{V} = \frac{\mathbf{V}' \cdot \omega}{|\omega|}\frac{\omega}{|\omega|} = \frac{\mathbf{V}' \cdot \omega}{\omega \cdot \omega}\omega, \tag{32}$$

Fig. 8. A body is moving with angular velocity $\boldsymbol{\omega}$, and point **Q** in the body is moving with linear velocity **V′**. A. The components of **V′** in the direction of $\boldsymbol{\omega}$ and perpendicular to $\boldsymbol{\omega}$ are shown. B. The shortest vector from the ISA to point **Q** is shown.



where

$$|\boldsymbol{\omega}| = \sqrt{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}.$$

Now the linear velocity along the *ISA*, the angular velocity about the *ISA*, and the direction of the *ISA* are known. Only a point on the *ISA* needs to be found to determine the moment, **M**, of the *ISA* about the origin. The second component of the linear velocity of **Q**, **N** in Figure 8A, is related to the angular velocity of the body and to the distance from the *ISA* to the point, **Q**, by the equation below (Fig. 8B):

$$\mathbf{N} = \boldsymbol{\omega} \times \mathbf{D}. \tag{33}$$

The vector **D** goes from the *ISA* to point **Q** and is perpendicular to the *ISA*, as shown in Figure 8B. Hence, by determining **D**, a point on the *ISA* will be found. In Figure 8A, **D** points out of the paper and **N**, $\boldsymbol{\omega}$, and **D** are mutually perpendicular. The unit vector in the direction of **D** is found as follows:

$$\frac{\mathbf{D}}{|\mathbf{D}|} = \frac{\mathbf{N}}{|\mathbf{N}|} \times \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}.$$

The vector **D** is found by rearranging this equation:

$$\mathbf{D} = \frac{|\mathbf{D}|}{|\mathbf{N}||\boldsymbol{\omega}|} \mathbf{N} \times \boldsymbol{\omega}.$$

From Eq. (33), the definition of the cross-product and the fact that $\boldsymbol{\omega}$ and **D** are perpendicular the following equation can be written.

$$|\mathbf{N}| = |\boldsymbol{\omega}||\mathbf{D}|.$$

Eliminate the magnitudes of **N** and **D** from the above equations.

$$\mathbf{D} = \frac{\mathbf{N} \times \boldsymbol{\omega}}{|\boldsymbol{\omega}||\boldsymbol{\omega}|}, \qquad \mathbf{D} = \frac{\mathbf{N} \times \boldsymbol{\omega}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}. \tag{34}$$

Another relationship involving **N** results from Figure 8A.

$$\mathbf{N} = \mathbf{V}' - \mathbf{V}. \tag{35}$$

Equation (35) is substituted into Eq. (34) to get an equation for **D** in terms of known quantities.

$$\mathbf{D} = \frac{(\mathbf{V}' - \mathbf{V}) \times \boldsymbol{\omega}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}.$$

This equation is simplified.

$$\mathbf{D} = \frac{(\mathbf{V}' \times \boldsymbol{\omega}) - (\mathbf{V} \times \boldsymbol{\omega})}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}.$$

Since **V** and $\boldsymbol{\omega}$ are parallel, the second term on the right vanishes.

$$\mathbf{D} = \frac{\mathbf{V}' \times \boldsymbol{\omega}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}. \tag{36}$$

Now equations for the **s** and **M** vectors can be written.

$$\mathbf{s} = \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}, \tag{37}$$

$$\mathbf{M} = (\mathbf{Q} - \mathbf{D}) \times \mathbf{s}. \tag{38}$$

This completes the determination of the *ISA* and **V** from the angular velocity of the body and the linear velocity of an arbitrary point in the body. These equations are independent of a coordinate system. The *ISA* and **V** found are referenced to the same coordinate system that the **V**′ and ω are referenced to. Equations (24) and (29) with Eq. (23) and either Eqs. (27) or (28) answer the second question cited in the introduction; given the position, orientation, and velocity of the platform, what are the necessary actuator velocities?

Here is an example to illustrate the use of the above equations. This example is an extension of the one used above. The angular velocity, ω, of the platform is [0 0 1] radians per second and the linear velocity, **V**′, of the point in the platform that is currently at [3 5 4] in the base coordinate system is [0 0 3] units per second. The velocity, **V**, along the *ISA* and the vector, **D**, from the *ISA* to the point are calculated as follows. Since the given linear velocity is parallel with the given angular velocity, the point for which the linear velocity is given must be on the *ISA* as verified below.

$$\mathbf{V} = \frac{[0\quad 0\quad 3]\cdot[0\quad 0\quad 1]}{[0\quad 0\quad 1]\cdot[0\quad 0\quad 1]}[0\quad 0\quad 1], \qquad (32')$$

$$\mathbf{V} = [0\quad 0\quad 3]$$

$$\mathbf{D} = \frac{[0\quad 0\quad 3]\times[0\quad 0\quad 1]}{[0\quad 0\quad 1]\cdot[0\quad 0\quad 1]} = 0. \qquad (36')$$

The **s** and **M** parts of the Plücker coordinate vector for the *ISA* are calculated as follows:

$$\mathbf{s} = \frac{1}{1}[0\quad 0\quad 1] = [0\quad 0\quad 1], \qquad (37')$$

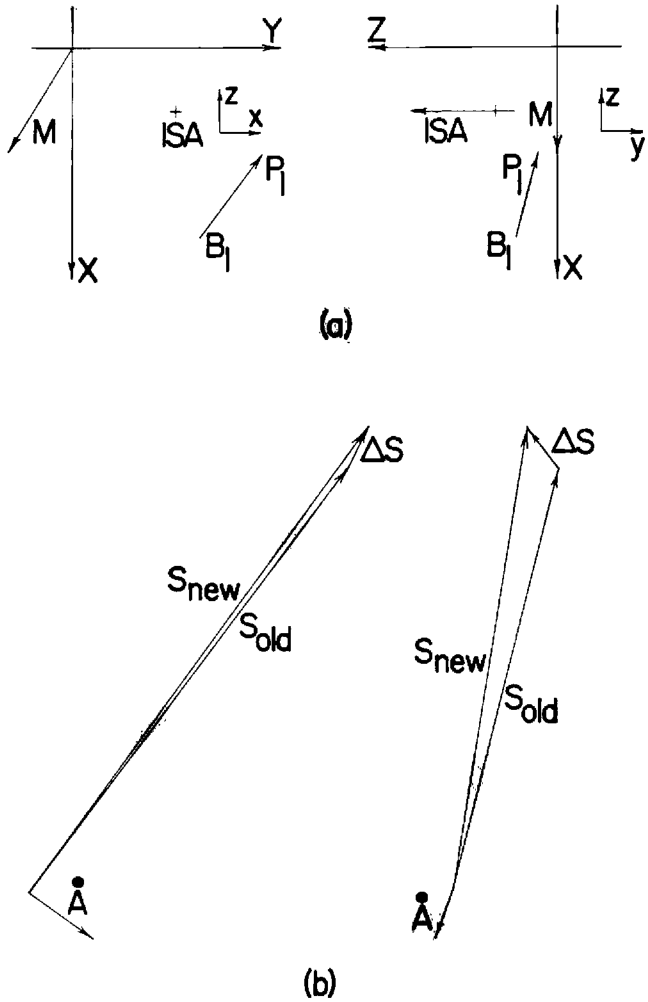
$$\mathbf{M} = [3\quad 5\quad 4]\times[0\quad 0\quad 1] = [5\quad -3\quad 0]. \quad (38')$$

Figure 9A shows the configuration. Next, the vector rate of change is calculated.

$$\dot{\mathbf{S}}_1 = [0\quad 0\quad 3] + [0\quad 0\quad 1] \times \left\{ [2\quad -3\quad -1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} + [4\quad 7\quad -2] - [0\quad 0\quad 1]\times[5\quad -3\quad 0] \right\}, \qquad (23')$$

$$\dot{\mathbf{S}}_1 = [-4\quad 2\quad 3].$$

Fig. 9. A. The platform is translating along and rotating about the ISA. *The moment shown is the moment of the* ISA *about the origin of the base coordinate system.* *B. The scale is expanded by a factor of five to show the change in the vector from* $\mathbf{B}_1$ *to* $\mathbf{P}_1$ *as the platform moves along and about the* ISA.



The rate of change of length of the leg is determined below:

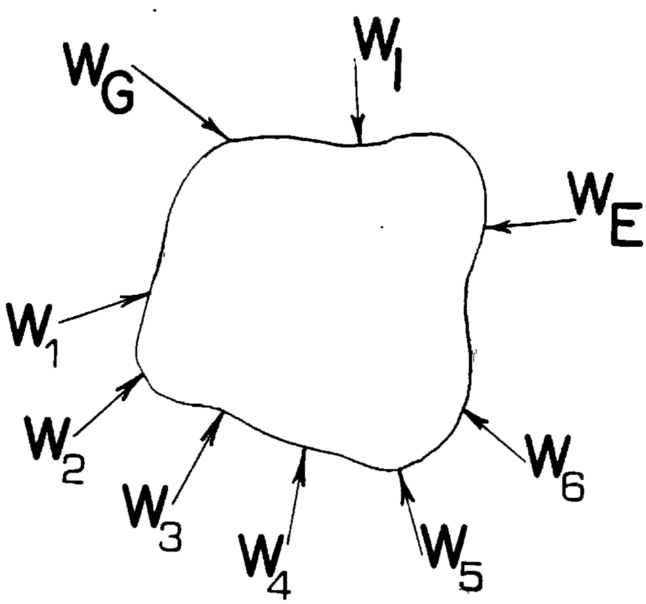
$$\dot{\sigma}_1 = [-4\quad 2\quad 3]\cdot[-0.78\quad 0.59\quad -0.20], \quad (24')$$

$$\dot{\sigma}_1 = 3.70 \qquad \text{units/second.}$$

The current position of point $\mathbf{P}_1$ is [5 9 1] in the base coordinate system. If the velocity, $\dot{\mathbf{S}}_1$, above is continued for 0.1 seconds, the position of $\mathbf{P}_1$ would be [4.6 9.2 1.3] and the change in length of the leg would be given by the following calculation:

$$\sqrt{(4.6-9)^2+(9.2-6)^2+(1.3-2)^2} - 5.1 = 0.39.$$

Fig. 10. The nine wrenches acting on the platform.



This checks, within round-off error, with the value of $\dot{\sigma}_1$ arrived at above.

The rate of change of the direction of the leg vector is determined below:

$$\dot{\mathbf{s}}_1 = [-4 \quad 2 \quad 3] - (3.70)[-0.78 \quad 0.59 \quad -0.20], \tag{25'}$$

$$\dot{\mathbf{s}}_1 = [-1.11 \quad -0.18 \quad 3.74],$$

$$\dot{\mathbf{A}}_1 = \frac{1}{5.1}([-0.78 \quad 0.59 \quad -0.20] \times [-1.11 \quad 0.18 \quad 3.74]), \tag{26'}$$

$$\dot{\mathbf{A}}_1 = [0.43 \quad 0.62 \quad -0.16].$$

Figure 9B shows an enlargement of the original leg vector and the leg vector after 0.1 seconds of the velocity in Eq. (23'). The figure also shows the $\dot{\mathbf{A}}_1$ vector.

This completes the development of the kinematic equations for the general Stewart Platform.

3. Dynamics of the General Stewart Platform

The reason for examining the dynamics of a manipulator is to determine the force or torque required of the actuators to balance the inertia forces and any forces applied to the manipulator by the external world. The dynamic analysis presented here is based on screw theory as presented by Hunt (1978). Hunt develops the concept of a wrench consisting of a force acting along some line and a couple acting about the same line. The force and the couple are related by the following equation, where h' is the pitch of the wrench and the magnitude of the force, f, is the intensity of the wrench.

$$\mathbf{C} = h'\mathbf{F}. \tag{39}$$

Figure 10 is a free body diagram of the platform. There are nine wrenches acting on the platform; six are exerted by the legs, one by the gravity field, one by the inertia load, and the last is exerted by the external world. The wrench exerted by the gravity field has pitch equal to zero (there is no couple) since the field is uniform. This paper will assume that the wrenches exerted by the legs also have zero pitch, which implies that screw joints are not used in the legs.

To ease the development of the dynamic equations, assume that the legs are massless. The sum of the force parts of the wrenches acting on the platform must equal zero.

$$^p\mathbf{F}_I + {^p\mathbf{F}_G} + {^p\mathbf{F}_E} + \sum_{i=1}^{6} {^p\mathbf{F}_i} = 0.$$

The inertia force on the platform may be replaced by minus the mass times the acceleration, and the gravity force may be replaced by the acceleration due to the gravity, $^p\mathbf{g}$, times the mass.

$$^p\mathbf{F}_E + \sum_{i=1}^{6} {^p\mathbf{F}_i} = m{^p\mathbf{a}} - m{^p\mathbf{g}}.$$

If the external force is moved to the right side of the equation, the left side can be written as a matrix product

$$^p\mathbf{s}^*\mathbf{f}^T = m({^p\mathbf{a}} - {^p\mathbf{g}}) - {^p\mathbf{F}_E}, \tag{40}$$

where $^p\mathbf{s}$ is the matrix of unit vectors along the legs in the platform coordinate system and $\mathbf{f}$ is a vector made up of the intensities of the wrenches exerted by the legs. This equation can be expanded.

$$\begin{bmatrix} {}^pS_{1x} & {}^pS_{2x} & \cdots & {}^pS_{6x} \\ {}^pS_{1y} & {}^pS_{2y} & \cdots & {}^pS_{6y} \\ {}^pS_{1z} & {}^pS_{2z} & \cdots & {}^pS_{6z} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} m({}^pa_x - {}^pg_x) - {}^pF_{Ex} \\ m({}^pa_y - {}^pg_y) - {}^pF_{Ey} \\ m({}^pa_z - {}^pg_z) - {}^pF_{Ez} \end{bmatrix}. \tag{41}$$

A similar development is done for the moment balance equation.

$${}^p\mathbf{C}_I + {}^p\mathbf{C}_E + \sum_{i=1}^{6} {}^p\mathbf{C}_i = 0,$$

$$\sum_{i=1}^{6} {}^p\mathbf{C}_i = {}^p\mathbf{C}_I - {}^p\mathbf{C}_E,$$

$${}^p\mathbf{M}^* \mathbf{f}^T = {}^p\mathbf{C}_I - {}^p\mathbf{C}_E, \tag{42}$$

where ${}^p\mathbf{M}^*$ is the matrix of moment vectors about the origin of the platform coordinate system. The sign of ${}^p\mathbf{C}_I$ results from the sign convention used here; this term can be expanded as follows:

$${}^p\mathbf{C}_I = -({}^p\mathbf{I} \quad {}^p\dot{\omega}^T) - \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} {}^p\mathbf{I}\, {}^p\omega^T.$$

$$\begin{bmatrix} {}^pM_{1x} & {}^pM_{2x} & \cdots & {}^pM_{6x} \\ {}^pM_{1y} & {}^p\mathrm{M}_{2y} & \cdots & {}^pM_{6y} \\ {}^pM_{1z} & {}^pM_{2z} & \cdots & {}^pM_{6z} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} -{}^pC_{Ix} - {}^pC_{Ex} \\ -{}^pC_{Iy} - {}^pC_{Ey} \\ -{}^pC_{Iz} - {}^pC_{Ez} \end{bmatrix} \tag{43}$$

Equations (40) and (42) can be combined into a single equation by using the Plücker coordinates of the leg vectors.

$${}^p\mathbf{U}\, \mathbf{f}^T = \begin{bmatrix} m({}^Pa_x - {}^Pg_x) - {}^PF_{Ex} \\ m({}^Pa_y - {}^Pg_y) - {}^PF_{Ey} \\ m({}^Pa_z - {}^Pg_z) - {}^PF_{Ez} \\ -{}^pC_{Ix} - {}^pC_{Ex} \\ -{}^pC_{Iy} - {}^pC_{Ey} \\ -{}^pC_{Iz} - {}^pC_{Ez} \end{bmatrix}. \tag{44}$$

All terms on the right side of Eq. (44) are known, as is the matrix of Plücker coordinates. The forces exerted by the legs can be solved for by inverting the 6×6 matrix of Plücker coordinates. An example of the use of this equation will be given later in this paper.

This solution for the forces exerted by the legs has assumed massless legs and has also assumed that the legs exert pure forces. Relaxation of these limits will be discussed in a future publication. Equation (44) is a partial answer to the third question cited in the introduction; given the position, orientation, and external force and torque on the platform, what are the necessary actuator forces?

4. Singular Positions of the Manipulator

Every mechanism has singular positions in which the general equations for the motion of the mechanism do not hold. In series-connected manipulators, the singular positions result in the loss of one or more degrees of freedom. In parallel-connection manipulators, the singular positions result in the gain of one or more degrees of freedom.

The singular positions of the Stewart Platform are determined from a screw system analysis, as described by Hunt (1978). The platform is kinematically constrained by six wrenches exerted by the six legs. As indicated above, the discussion in this paper will be limited to mechanisms in which the legs exert pure forces. To successfully constrain the platform, these six forces must act along lines that are linearly independent. When the six lines of action of the forces are linearly dependent, the platform is in a singular position and gains one or more degrees of freedom.

The search for singular positions has thus been re-

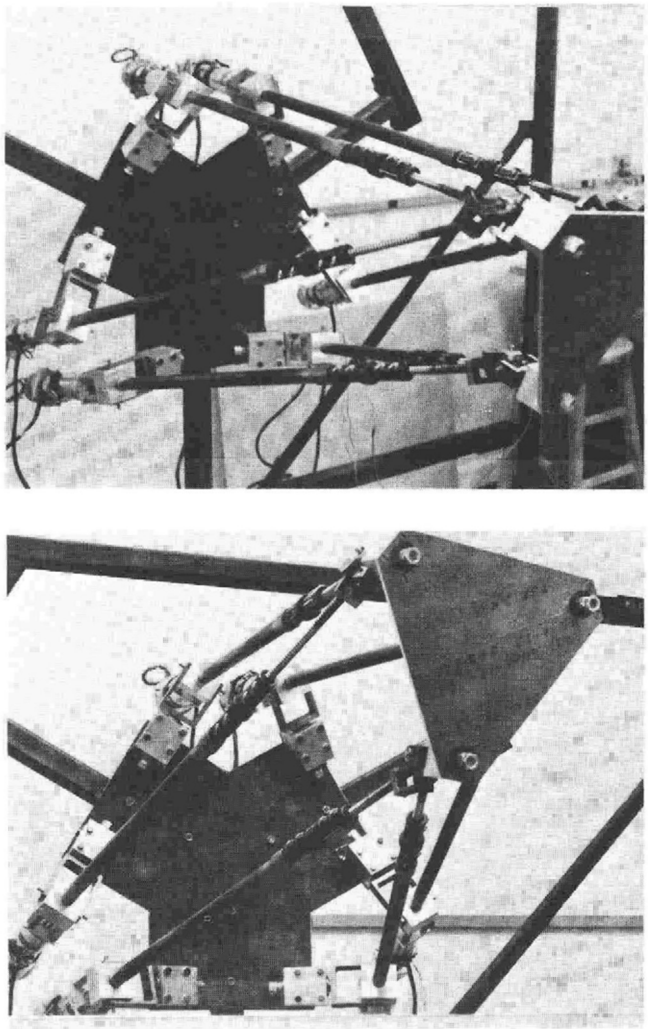


Fig. 11. Two views of the most recent Stewart Platform built at Oregon State University.

duced to the search for positions in which the six leg vectors are not linearly independent. This condition can be tested for by checking for singularities of the matrix of the Plücker coordinates, **U**, of the six legs. If the determinant of this matrix is zero, the manipulator is in a singular position.

The general solution to this problem is beyond the scope of this paper. Particular solutions to the singular position problem for the practical versions of the manipulator discussed later in this paper are presented below.

In a real manipulator, the results of this analysis are complicated by the other three wrenches acting on the platform and by the bearing clearances at the joints.

This provides a partial answer to the fourth question cited in the introduction — Is the manipulator singular for a particular position and orientation of the platform?

5. Design of a Practical Stewart Platform Manipulator

The equations derived above are for general positions of the ends of the legs in the base and the platform. The construction of a real machine requires that practical constraints be placed on this general case. Such a practical machine, which has actually been built, is described in the following paragraphs. The design aids and design procedures used in the construction of this machine are also discussed. Figure 11 shows some photographs of the most recent Stewart Platform built at Oregon State University.

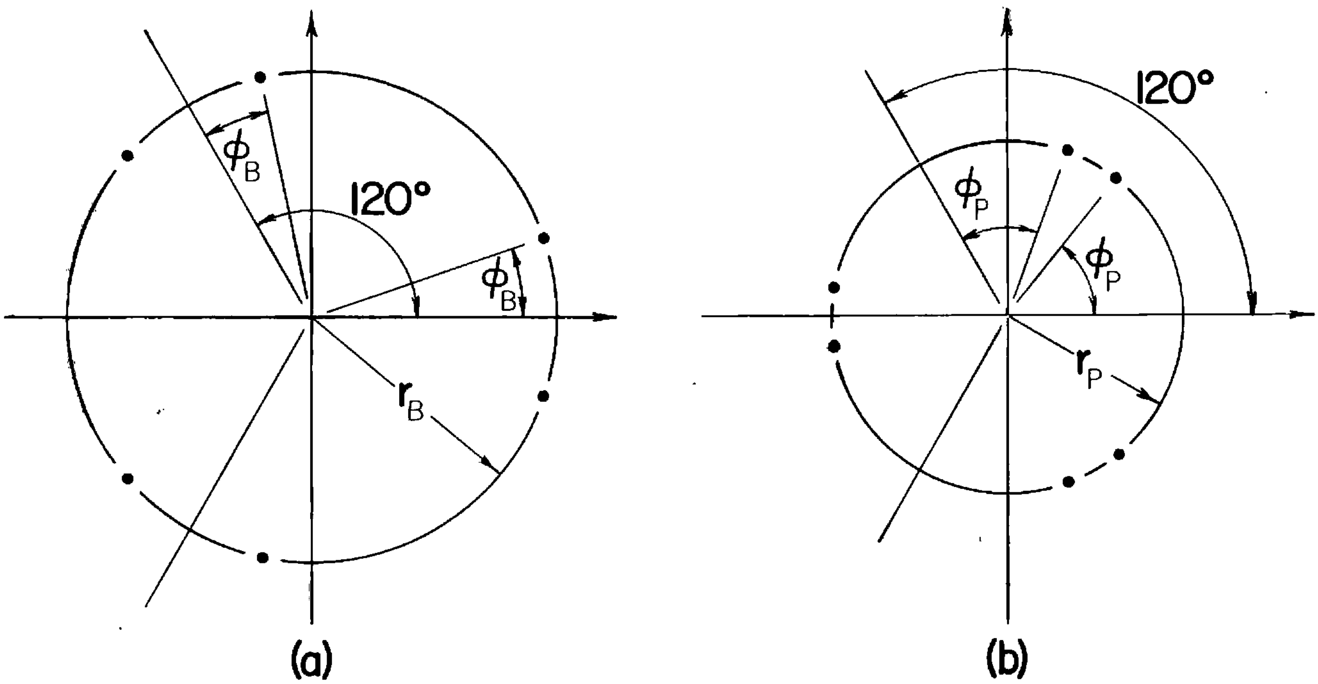
The ends of the legs in the base are arranged in a plane, as are the ends of the legs in the platform. The six points in each of these planes are arranged in semiregular hexagons, as shown in Fig. 12. If regular hexagons are used in both base and platform, the equations derived above do not hold; screw theory can be used to verify this fact. The planes in which these hexagons lie are the $z = 0$ planes in the base and platform coordinate systems. The vector representations of the six points in the base can now be written as follows: (Note that in the following equations the terms in square brackets are the components of vectors.)

$$\mathbf{B}_i = \mathbf{r}_B[\mathrm{c}A_{Bi}\ \mathrm{s}A_{Bi}\ 0], \tag{45}$$

where

$$\begin{aligned}
A_{B1} &= \phi_B, \\
A_{B2} &= \frac{2}{3}\pi - \phi_B, \\
A_{B3} &= \frac{2}{3}\pi + \phi_B, \\
A_{B4} &= -\left(\frac{2}{3}\pi + \phi_B\right), \\
A_{B5} &= -\left(\frac{2}{3}\pi - \phi_B\right), \\
A_{B6} &= -\phi_B.
\end{aligned} \tag{46}$$

Fig. 12. Simplified Stewart Platform. A. The six points in the base. B. The six points in the platform.



In the above equation, the c and s prefixes of the angles stand for cosine and sine respectively. Vector representations of the six points in the platform can be written in a similar way.

$$^{P}\mathbf{P}_i = \mathbf{r}_P[\mathrm{c}A_{Pi}\ \mathrm{s}A_{Pi}\ 0], \tag{47}$$

where

$$\begin{aligned} A_{P1} &= \phi_P &&= \frac{\pi}{3}, \\ A_{P2} &= \frac{2}{3}\pi - \phi_P &&= \frac{\pi}{3}, \\ A_{P3} &= \frac{2}{3}\pi + \phi_P &&= \pi, \\ A_{P4} &= -\left(\frac{2}{3}\pi + \phi_P\right) &&= \pi, \\ A_{P5} &= -\left(\frac{2}{3}\pi - \phi_P\right) &&= -\frac{\pi}{3}, \\ A_{P6} &= -\phi_P &&= -\frac{\pi}{3}. \end{aligned} \tag{48}$$

This mechanism is further simplified by setting the angle ϕ_P to $\frac{\pi}{3}$ radians as on the right above. As Fig. 13 shows, this makes the points in the platform coincide in pairs.

5.1. Leg Vectors and Leg Moments

This arrangement of points allows a certain amount of simplification in the leg vector and leg moment equations. The leg vector Eq. (11) can be rewritten as follows as three component scalar equations:

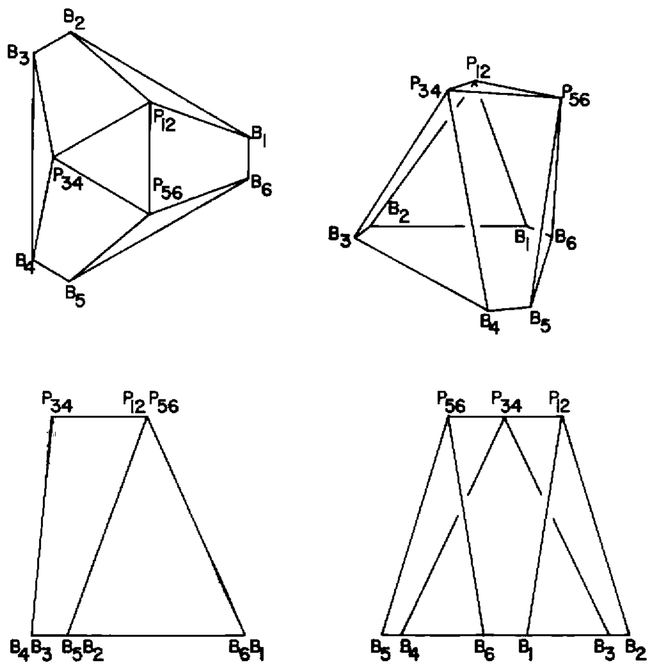
$$\begin{aligned} S_{ix} &= r_p(\alpha_x \quad \mathrm{c}A_{Pi} + \beta_x \quad \mathrm{s}A_{Pi}) + T_x - r_B \quad \mathrm{c}A_{Bi}, \\ S_{iy} &= r_p(\alpha_y \quad \mathrm{c}A_{Pi} + \beta_y \quad \mathrm{s}A_{Pi}) + T_y - r_B \quad \mathrm{c}A_{Bi}, \\ S_{iz} &= r_p(\alpha_z \quad \mathrm{c}A_{Pi} + \beta_z \quad \mathrm{s}A_{Pi}) + T_z. \end{aligned} \tag{49}$$

The leg moment Eq. (14) can be rewritten as follows:

$$\mathbf{M}_i = r_B[\mathrm{c}A_{Bi} \quad \mathrm{s}A_{Bi} \quad 0] \times \frac{1}{\sigma_i}[S_{ix} \quad S_{iy} \quad S_{iz}],$$

$$\mathbf{M}_i = \frac{r_B}{\sigma_i}([\mathrm{c}A_{Bi} \quad \mathrm{s}A_{Bi} \quad 0] \times [S_{ix} \quad S_{iy} \quad S_{iz}]),$$

Fig. 13. The kinematic arrangement of the simplified Stewart Platform with triangular platform is shown in top, front, right side, and pictorial views.



$$\mathbf{M}_i = \frac{r_B}{\sigma_i}[(S_{iz} \quad sA_{Bi})(-S_{iz} \quad cA_{Bi}) \\ (S_{iy} \quad cA_{Bi} - S_{ix} \quad sA_{Bi})].$$

This equation can be expanded into three scalar equations as follows:

$$M_{ix} = \frac{r_B \, sA_{Bi}}{\sigma_i}\{r_p(\alpha_z \quad cA_{pi} + \beta_z \quad sA_{Pi}) + T_z\},$$

$$M_{iy} = \frac{-r_B \, cA_{Bi}}{\sigma_i}\{r_P(\alpha_z \quad cA_{Pi} + \beta_z \quad sA_{Pi}) + T_z\}, \tag{50}$$

$$M_{iz} = \frac{r_B \, cA_{Bi}}{\sigma_i}\{r_P(\alpha_y \quad cA_{Pi} + \beta_y \quad sA_{Pi}) + T_y\}, \\ -\frac{r_B \, sA_{Bi}}{\sigma_i}\{r_P(\alpha_x \quad cA_{Pi} + \beta_x \quad sA_{Pi}) + T_x\}.$$

The leg vector and leg moment equations referenced to the platform coordinate system also simplify. Equation (15) for the leg vector is rewritten below.

$$\begin{aligned} {}^PS_{ix} &= r_P \quad cA_{Pi} + (T_x - r_B \quad cA_{Bi})\alpha_x \\ &\quad + (T_y - r_B \quad sA_{Bi})\alpha_y + T_z \quad \alpha_z, \\ {}^PS_{iy} &= r_P \quad cA_{Pi} + (T_x - r_B \quad cA_{Bi})\beta_x \\ &\quad + (T_z - r_B \quad sA_{Bi})\beta_y + T_z \quad \beta_z, \\ {}^PS_{iz} &= (T_x - r_B \quad cA_{Bi})\gamma_x \\ &\quad + (T_y - r_B \quad sA_{Bi})\gamma_y + T_z \quad \gamma_z. \end{aligned}$$

This equation is further expanded below:

$$\begin{aligned} {}^PS_{ix} &= r_P \quad cA_{Pi} + T_x \quad \alpha_x + T_y \quad \alpha_y + T_z \quad \alpha_z \\ &\quad - r_B(\alpha_x \quad cA_{Bi} + \alpha_y \quad sA_{Bi}), \\ {}^PS_{iy} &= r_P \quad cA_{Pi} + T_x \quad \beta_x + T_y \quad \beta_y + T_z \quad \beta_z \\ &\quad - r_B(\beta_x \quad cA_{Bi} + \beta_y \quad sA_{Bi}), \\ {}^PS_{iz} &= T_x \quad \gamma_x + T_y \quad \gamma_y + T_z \quad \gamma_z - r_B(\gamma_x \quad cA_{Bi} \\ &\quad + \gamma_y \quad sA_{Bi}). \end{aligned} \tag{51}$$

Equation (19) for the leg moment is rewritten below.

$$\begin{aligned} {}^PM_{ix} &= \frac{r_P \, sA_{Pi}}{\sigma_i}\{T_x \quad \gamma_x + T_y \quad \gamma_y + T_z \quad \gamma_z \\ &\quad - r_B(\gamma_x \quad cA_{Bi} + \gamma_y \quad sA_{Bi})\}, \\ {}^PM_{iy} &= \frac{-r_P \, cA_{Pi}}{\sigma_i}\{T_x \quad \gamma_x + T_y \quad \gamma_y + T_z \quad \gamma_z \\ &\quad - r_B(\gamma_x \quad cA_{Bi} + \gamma_y \quad sA_{Bi})\}, \\ {}^PM_{iz} &= \frac{r_P \, cA_{Pi}}{\sigma_i}\{T_x \quad \beta_x + T_y \quad \beta_y + T_z \quad \beta_z \\ &\quad - r_B(\beta_x \quad cA_{Bi} + \beta_y \quad sA_{Bi})\} \\ &\quad - \frac{r_P \quad sA_{Pi}}{\sigma_i}\{T_x \quad \alpha_x + T_y \quad \alpha_y + T_z \quad \alpha_z \\ &\quad - r_B(\alpha_x \quad cA_{Bi} + \alpha_y \quad sA_{Bi})\}. \end{aligned} \tag{52}$$

5.2. Forces Exerted by Legs

What are the forces in the legs in this simplified Stewart Platform? Take the simple example with no rotation and translation only in the z-direction to see how the general equations presented above work. This is the position illustrated in Fig. 13.

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{T} = [\ 0 \quad 0 \quad T_z\ ].$$

With these values for rotation and translation, Eq. (51) for the leg vector and Eq. (52) for the leg moment simplify as follows:

$$^{P}S_{ix} = r_p \quad cA_{Pi} - r_B \quad cA_{Bi},$$

$$^{P}S_{iy} = r_P \quad sA_{Pi} - r_B \quad sA_{Bi},$$

$$^{P}S_{iz} = T_z,$$

$$^{P}M_{ix} = \frac{1}{\sigma} r_P \quad T_z \quad sA_{Pi},$$

$$^{P}M_{iy} = -\frac{1}{\sigma} r_P \quad T_z \quad cA_{Pi},$$

$$^{P}M_{iz} = \frac{1}{\sigma_i} r_B \quad r_P \quad s(A_{Pi} - A_{Bi}).$$

Make the further assumptions that there is no external force, that the platform is not accelerating, and that the force due to gravity is a vector in the platform $-z$ direction with unit magnitude. Then the leg force Eq. (44) is as follows:

$$\begin{bmatrix} \frac{1}{\sigma_1}(r_P cA_{P1} - r_B cA_{B1}) & \cdots & \frac{1}{\sigma_6}(r_P cA_{P6} - r_B cA_{B6}) \\ \frac{1}{\sigma_1}(r_P sA_{P1} - r_B sA_{B1}) & \cdots & \frac{1}{\sigma_6}(r_P sA_{P6} - r_B sA_{B6}) \\ \frac{1}{\sigma_1} T_z & \cdots & \frac{1}{\sigma_6} T_z \\ \frac{1}{\sigma_1} r_P T_z sA_{P1} & \cdots & \frac{1}{\sigma_6} r_P T_z sA_{P6} \\ -\frac{1}{\sigma_1} r_P T_z cA_{P1} & \cdots & -\frac{1}{\sigma_6} r_P T_z cA_{P6} \\ \frac{1}{\sigma_1} r_B r_P s(A_{P1} - A_{B1}) & \cdots & \frac{1}{\sigma_6} r_B r_P s(A_{P6} - A_{B6}) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For example, let $r_P = 5$, $r_B = 15$, $\sigma_B = 10°$, and $T_z = 20$. The above equation becomes:

$$\begin{bmatrix} -0.52 & 0.32 & 0.20 & 0.20 & 0.32 & -0.52 \\ 0.074 & -0.42 & -0.49 & 0.49 & 0.42 & -0.074 \\ 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 \\ 3.68 & 3.68 & 0 & 0 & -3.68 & -3.68 \\ -2.12 & -2.12 & 4.25 & 4.25 & -2.12 & -2.12 \\ 2.44 & -2.44 & 2.44 & -2.44 & 2.44 & -2.44 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

When this equation is solved for f_1–f_6, each of the leg forces is found to be 0.196. They must be all the same because of symmetry. This is the correct value since the vertical component of the force is 0.85×0.196, which multiplied by 6 equals 1.

5.3. Stewart Platform Leg Mechanism

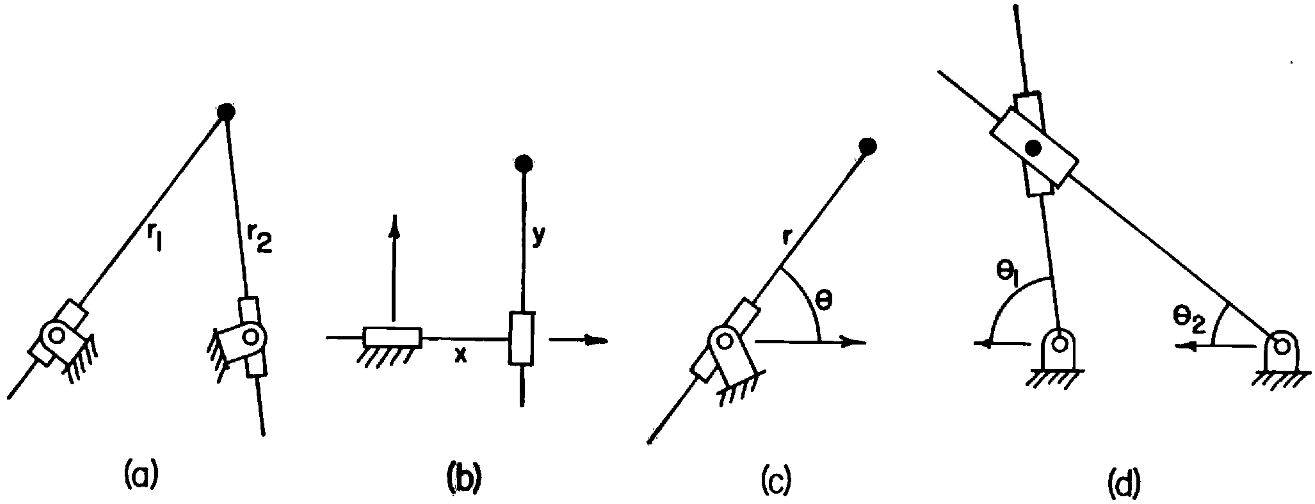
Up to this point, the Stewart Platform has been considered to be two bodies connected by six legs with each leg connected to each body by a ball-and-socket joint. Figure 13 shows the manipulator now being considered with six distinct points in the base and the six points in the platform coinciding in pairs. Notice that there are three triangles supporting the platform, $B_1B_2P_{12}$, $B_3B_4P_{34}$, and $B_5B_6P_{56}$. In each triangle the point P_{ij} can lie anywhere in the plane of the supporting triangle (within the mechanical limits). Each triangle can rotate about the line B_iB_j, allowing each point P_{ij} to be positioned anywhere in space.

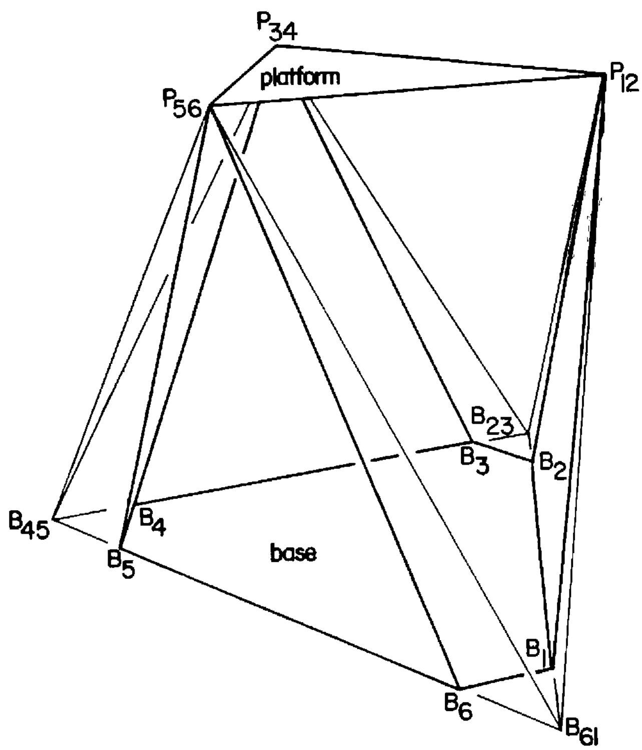
Since the points P_{ij} move in a plane, any mechanism that moves a point in a plane can be substituted for the triangle. Figure 14 shows some possibilities including the present one; the controlled variables are all labeled. With each of these possibilities except for the present one, there is the mechanical disadvantage of depending on bending moments for support. With the present arrangement, the platform load is supported by tension or compression of the legs.

A second implication of this view of the mechanism

Fig. 14. Four mechanisms that position a point in a plane. A. The one used in the Stewart Platform.

Fig. 15. The platform of a simplified Stewart Platform has the same motion as it would if the base of the mechanism were a triangle produced by extending the long sides of the semiregular hexagonal base.



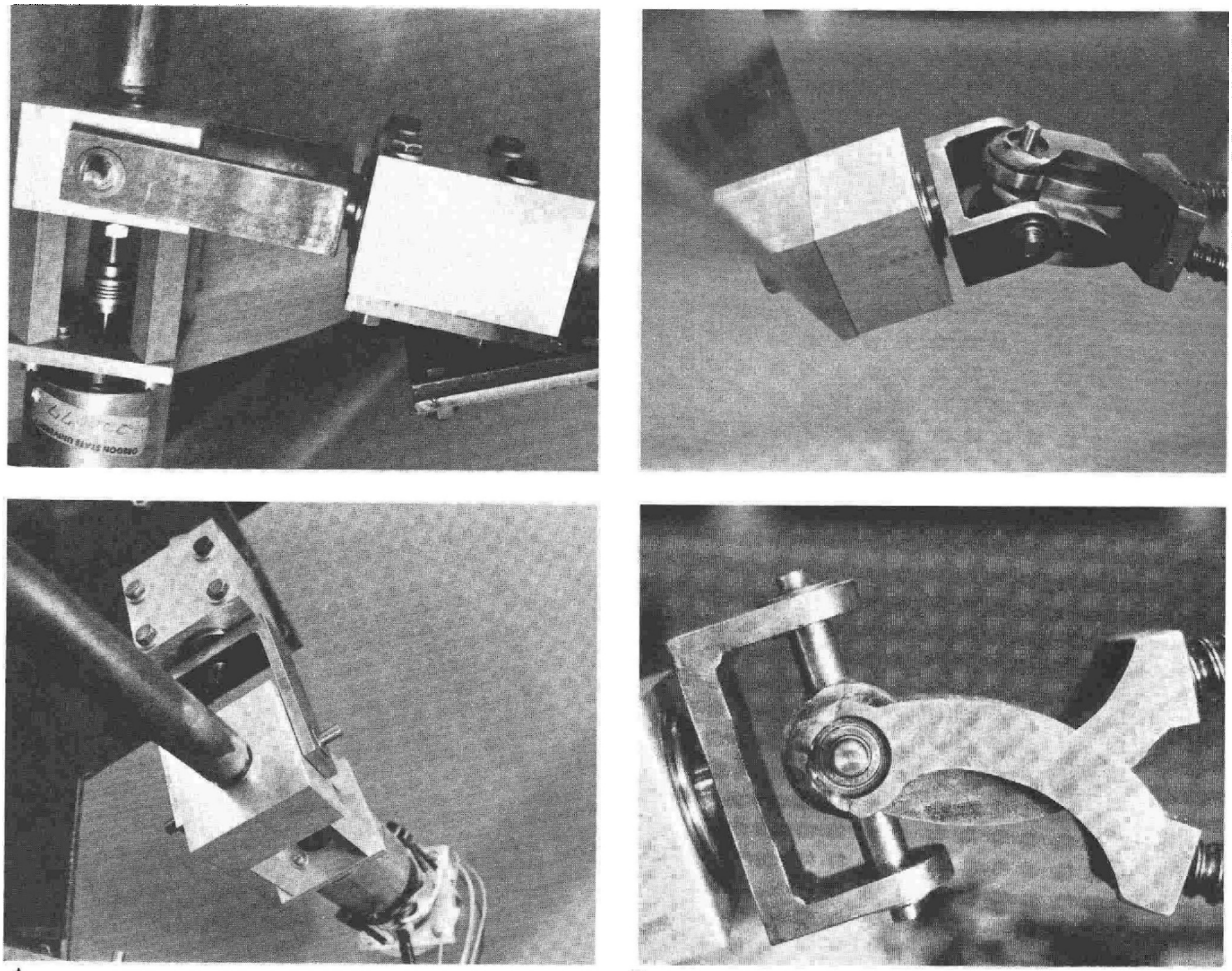


is illustrated in Fig. 15, where the lines B_iB_j have been extended to their intersections at points $\mathbf{B}_{mn}$. If the original supporting triangles were replaced with $B_{61}B_{23}P_{12}$, and so forth, the motion of the platform would not change since the points P_{ij} that define platform position and orientation still move in the same plane rotating about the same line in the base. However, the lengths, velocities, and forces required of the actuators would change.

The Stewart Platform manipulators built at Oregon State University have all used screw jacks driven by electric motors as leg actuators. This design has the advantage of being light and easy to control. Velocity and position control can be done using shaft encoders and tachometers. Friction can be minimized in a screw drive by using a ball screw, and the backlash in the ball screw can be eliminated by using double nuts preloaded with spring washers. It was noted above that, in general, using screw joints in the legs means that each leg will apply a couple to the platform. This is not true when adjacent legs are connected together to form triangles. In this case, each leg in a connected part counteracts the couple exerted by the other leg in the pair. The platform is effectively supported by six pure forces.

The ends of the legs should be mounted on gimbals (Hooke joints), not on ball-and-socket joints. If it is designed properly, a gimbal gives a much greater range of motion than a ball-and-socket joint. Figure 16 shows photographs of the base and platform gimbals on the last Stewart Platform built at Oregon State University. The platform gimbal is doubled to make the platform ends of two adjacent legs coincident. The platform gimbal has a third axis perpendicular to the platform plane, which makes it equivalent to a double

Fig. 16. Some views of the joints at the ends of the legs of the most recent Stewart Platform built at Oregon State University. A. Two views of the gimbal joint at the base. B. Two views of the gimbal joint at the platform. The gimbal joint at the platform consists of a double Hooke joint and a pivot perpendicular to the platform plane.



A B

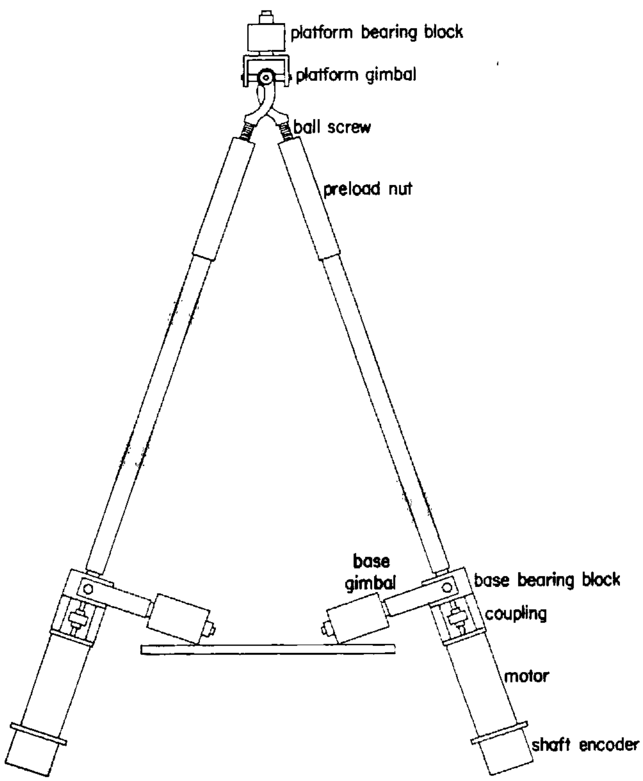
ball joint. The base gimbal has the first revolute axis inclined to the base plate to increase the useful range of motion of the joint.

Figure 17 illustrates one of the leg triangles. The leg pivot at the base, at the intersection of the two revolute axes of the gimbal, is near the average center of mass of the leg/motor combination. The leg was designed in this way to minimize the bending of the leg due to its own weight.

5.4. Determining Range of Motion

A simulation of the Stewart Platform has been written to investigate some of its kinematics. The limits used to find the range of motion of the platform in this simulation are the maximum and minimum lengths of the legs and a limit on how close the platform hexagon is allowed to come to the base plane. No check is included for singular positions of the manipulator.

Fig. 17. The leg triangle of the most recent Stewart Platform built at Oregon State University.



One of the two outputs of the simulation is a plot of a cross section of the work envelope of the manipulator. These plots are useful for developing an understanding of the size and shape of the work envelope. To make one of these plots, any four of the six parameters, x, y, z, roll, pitch, yaw, of the motion of the platform are held constant while the other two vary. The most useful results are obtained when roll, pitch, yaw, and one of the others are held constant; by doing this front, side, and top views of the work envelope can be drawn. The simulation has the capability of plotting multiple cross sections on parallel cutting planes, thus producing a contour map of the work envelope. Some examples of this output are shown in Fig. 18.

After the plane to work in has been chosen, the simulation works by finding a point on the boundary of the work envelope and then following the boundary. The method of following the boundary is a variation on the method described by Mason (1956) and Cordray (1957).

The other output of the simulation is the extremes of movement of each one of the joints of any one of the six legs. For any position of the platform, the position and orientation of the final joint in each leg may be found by a simple transformation. The values for each of the joint variables for a leg are then determined by applying the conventional series manipulator solution. This simulation output has been very useful in designing and orienting joints. For instance, the depth of the yoke on the base gimbal and the angle at which the base gimbal is set were determined in this way. Another example is the curve in the yokes for the platform gimbal.

This simulation allows the designer to first choose a range of motion for the platform from a catalog or an interactive run of the simulation. The results of this first part of the simulation are size and shape of platform and base planes. Then the joint range of motion part of the simulation is run to facilitate joint sizing and orientation.

5.5. Singular Positions

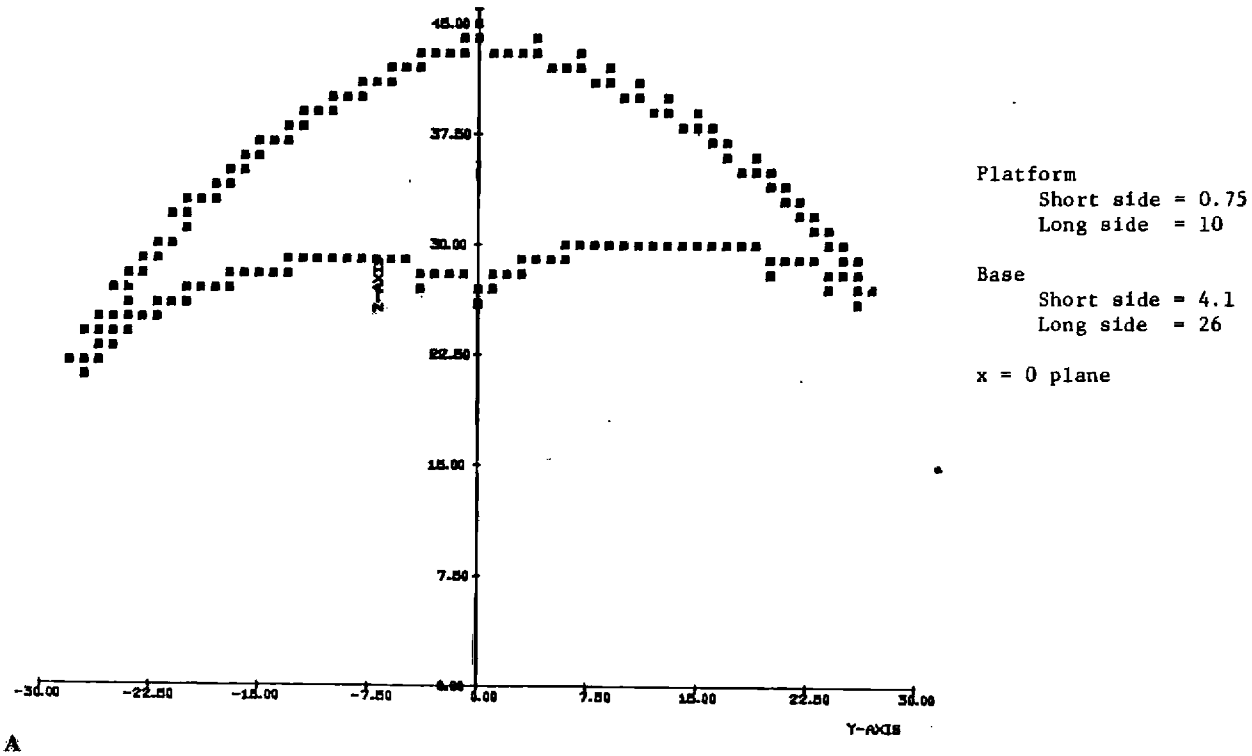
When experimenting with a model Stewart Platform manipulator, it is not difficult to find two apparently different varieties of singular position as illustrated in Figs. 19 and 20. In Fig. 19, all three points on the platform and the points in the base at the other two corners of one of the supporting triangles lie in a plane. The six leg vectors all intersect one line, line $P_{56}P_{12}$ in Fig. 19, a configuration that Hunt (1978) has shown to be a special case always resulting in linear dependence of the six leg vectors. The degree of freedom gained is a pure rotation about the line that all six leg vectors intersect.

This singular position of the mechanism is reached by a rotation about the y-axis. When the platform is a triangle, the angle of this rotation can be obtained from the geometry, as shown in Fig. 21. The equation follows:

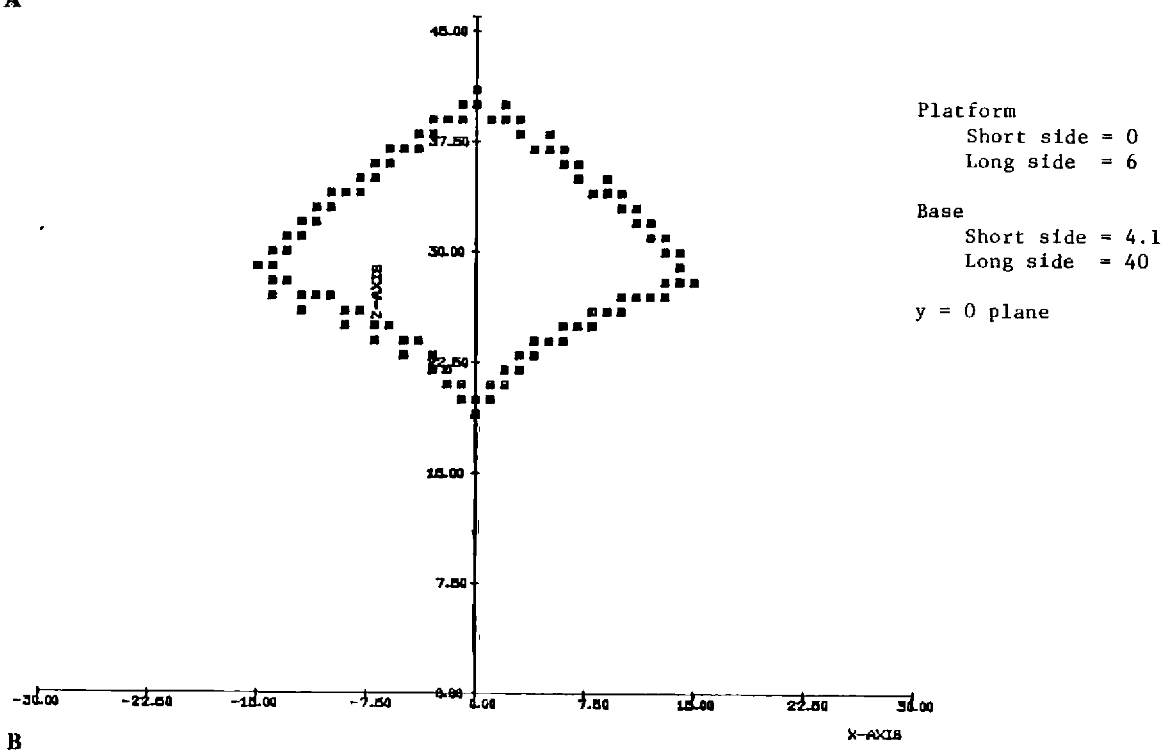
$$R_y = \arctan \frac{-T_z}{T_x - r_B cA_{B3}}.$$

In this equation, the rotation about the y-axis depends only on the translation in the x and z directions.

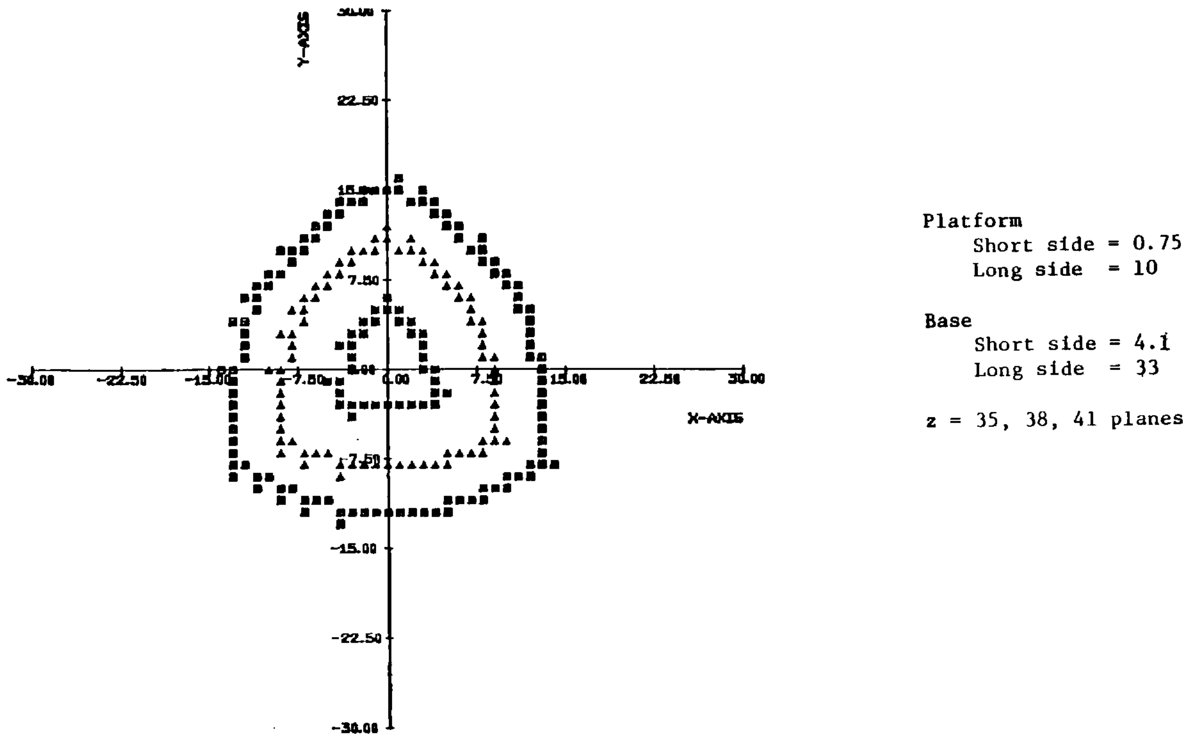
Fig. 18. Three sample plots of the work envelope produced by the simulation of the Stewart Platform. For each of the three samples, the rotation, **R**, *is the identity. The other parameters are as follows. A. Platform short side = 0.75; platform long side = 10; base short side = 4.1; base long side = 26; plane: x = 0. B. Platform short side = 0; platform long side = 6; base short side = 4.1; base long side = 40; plane: y = 0. C. Platform short side = 0.75; platform long side = 10; base short side = 4.1; base long side = 33; plane: z = 35, 38, 41.*



A



B



C

The y-translation is arbitrary, and the rotation of the platform about the platform z-axis is also arbitrary. Changes in these two parameters leave the six leg vectors intersecting the same line.

The equation above specifies a rotation about the y-axis to reach this singular position. From symmetry, there are two other lines that can be rotated about to reach similar singular positions. These require some combination of rotations about the x-axis and the y-axis.

The degree of freedom gained in the singular position illustrated in Fig. 20 is a screw motion of the platform about the platform z-axis. Examine the 6×6 matrix of Plücker coordinates of the leg lines to determine the characteristics of this singularity. The rotation and translation are given below:

$$\mathbf{R} = \begin{bmatrix} cR_z & -sR_z & 0 \\ sR_z & cR_z & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{T} = [\ T_x \quad T_y \quad T_z].$$

The unnormalized Plücker coordinates are written.

$$\begin{aligned}
S_{ix} &= r_P(cR_z cA_{Pi} - sR_z sA_{Pi}) + T_x - r_B cA_{Bi}, \\
S_{iy} &= r_P(sR_z cA_{Pi} - cR_z sA_{Pi}) + T_y - r_B sA_{Bi}, \\
S_{iz} &= T_z, \\
M'_{ix} &= r_B T_z sA_{Bi}, \\
M'_{iy} &= -r_B T_z cA_{Bi}, \\
M'_{iz} &= r_B cA_{Bi}\{r_P(sR_z cA_{Pi} + cR_z sA_{Pi}) + T_y\} \\
&\quad - r_B sA_{Bi}\{r_P(cR_z cA_{Pi} - sR_z sA_{Pi}) + T_z\}.
\end{aligned}$$

These equations are simplified as follows:

$$\begin{aligned}
S_{ix} &= r_P c(A_{Pi} + R_z) + T_x - r_B cA_{Bi}, \\
S_{iy} &= r_P s(A_{Pi} + R_z) + T_y - r_B sA_{Bi}, \\
S_{iz} &= T_z, \\
M'_{ix} &= r_B T_z sA_{Bi}, \\
M'_{iy} &= -r_B T_z cA_{Bi}, \\
M'_{iz} &= r_B r_P s(A_{Pi} - A_{Bi} + R_z) + r_B T_y cA_{Bi} - r_B T_x sA_{Bi}.
\end{aligned}$$

These six equations each form a row of the matrix as i varies from 1 to 6. Since the third row of the matrix is constant, another constant row will make the determi-

Fig. 19. The top, front, and right side views and a pictorial view of the Stewart Platform in a singular position. In this position, the platform has gained an extra degree of freedom, which is a pure rotation about line $P_{56}P_{12}$.

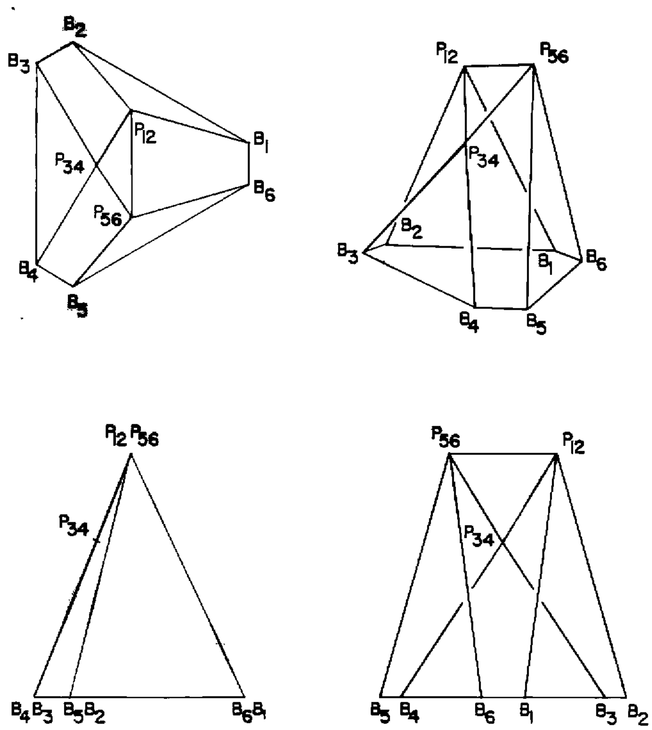
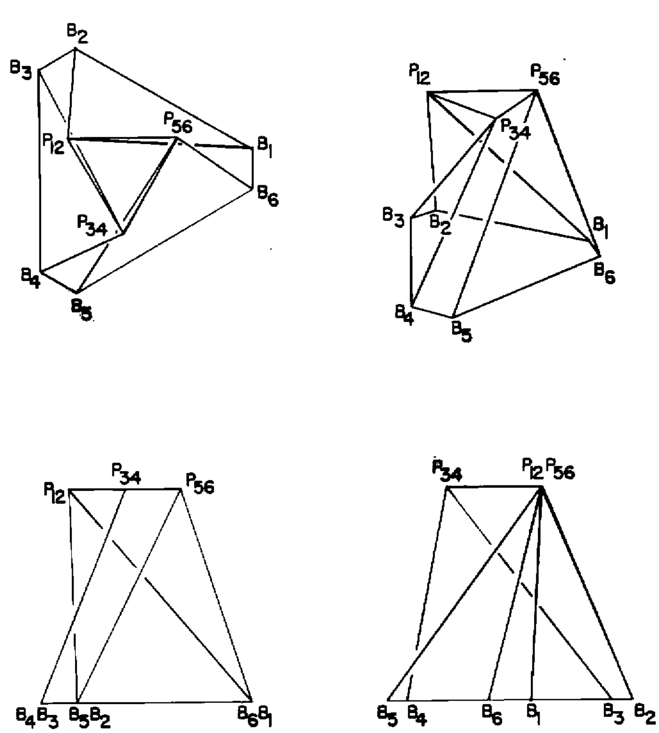


Fig. 20. The top, front, and right side views and a pictorial view of the Stewart Platform in a singular position. In this position, the platform has gained an extra degree of freedom, which is a screwing motion about the platform z-axis.



nant zero. The last row is replaced with the following linear combination of rows without changing the value of the determinant:

$$M'_{iz} + \frac{T_x}{T_z} M'_{ix} + \frac{T_y}{T_z} M'_{iy}.$$

Substitute for M'_{ix}, M'_{iy}, and M'_{iz} from the above equations and simplify.

$$r_B r_P\{s(A_{Pi} - A_{Bi})cR_z + c(A_{Pi} - A_{Bi})sR_z\}.$$

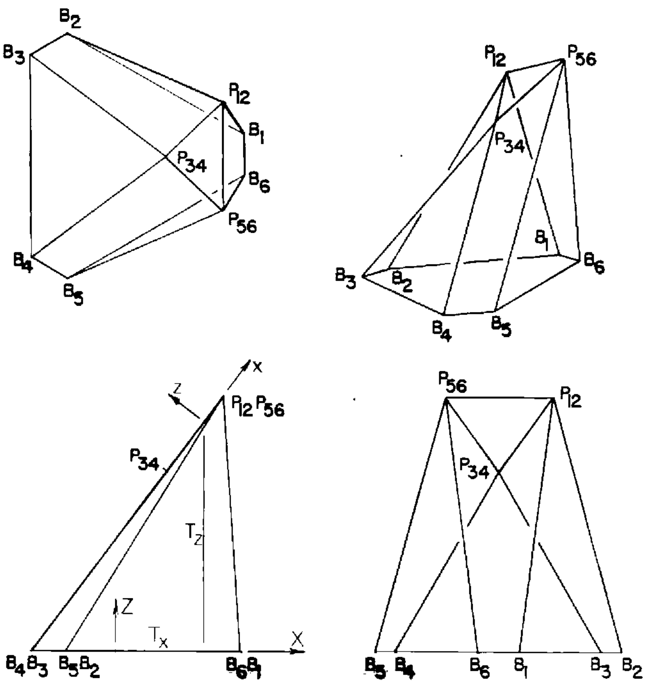
What are the values of $A_{Pi} - A_{Bi}$ as i varies from 1 to 6?

A_{Pi}	A_{Bi}	$A_{Pi} - A_{Bi}$
ϕ_P	ϕ_B	$\phi_P - \phi_B$
$2/3\pi - \phi_P$	$2/3\pi - \phi_B$	$-(\phi_P - \phi_B)$
$2/3\pi + \phi_P$	$2/3\pi + \phi_B$	$\phi_P - \phi_B$
$-(2/3\pi + \phi_P)$	$-(2/3\pi + \phi_B)$	$-(\phi_P - \phi_B)$
$-(2/3\pi - \phi_P)$	$-(2/3\pi - \phi_B)$	$\phi_P - \phi_B$
$-\phi_P$	$-\phi_B$	$-(\phi_P - \phi_B)$

The factor $c(A_{Pi} - A_{Bi})$ is constant for all six values of i, but the factor $s(A_{Pi} - A_{Bi})$ is not constant. Thus, the last row of the matrix as modified is constant if, and only if, R_z has the value $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$. This result does not make use of the restrictions imposed by having the platform a triangle; it is true for the semi-regular hexagon platform.

According to this result, if the platform is positioned anywhere and rotated one quarter turn in either direction about the platform z-axis while maintaining its plane parallel to the base plane, then the manipulator is in a singular position. This is easy to verify by building a paper model, which can be done by making eight triangles of the appropriate shape and fastening them together with tape. The resulting model will move a small amount. This is an interesting verification of the usefulness of screw theory since the common wisdom says that closed forms built only of triangles are rigid.

Fig. 21. The top, front, and right side views and a pictorial view of the Stewart Platform in a singular position. The derivation of the formula for this singular position is illustrated.



6. Conclusions

This paper presents the necessary equations for use of the Stewart Platform as a robot manipulator. Equations for the completely general case and for a special, more practical, case have been discussed. This has been a completely theoretical presentation in the sense that no consideration has been given to the real-time solution of the equations.

The Stewart Platform appears to have some advantages over more conventional manipulator designs in certain situations (Fichter and McDowell 1980). The equations presented here allow control of position, velocity, and force.

There are several areas needing further work. Among these are force control, dynamics, and singular positions. These topics will be discussed in greater depth in future publications.

Acknowledgments

Mr. Fernando Terracena did the coding and some of the programming of the simulation. I would like to thank Professor Kenneth Hunt and Professor Joe Davidson for the discussions we have had on screw theory. I would also like to thank Dr. Edward McDowell, who encouraged me to work on the Stewart Platform.

Appendix

$\dot{A}_i$ angular velocity of leg i; referenced to the base coordinate system

${}^{P}\dot{A}_i$ angular velocity of leg i; referenced to the platform coordinate system

A_{Bi} angle from the x-axis of the base coordinate system to the line connecting the origin to point B_i

A_{Pi} angle from the x-axis of the platform coordinate system to the line connecting the origin to point P_i

${}^{P}\mathbf{a}$ vector acceleration of the platform; referenced to the platform coordinate system

$\mathbf{B}_i$ vector description of the end of leg i in the base; referenced to the base coordinate system

${}^{P}\mathbf{B}_i$ vector description of the end of leg i in the base; referenced to the platform coordinate system

$\mathbf{C}$ couple vector

${}^{P}\mathbf{C}_E$ couple vector exerted on the platform by the external world; referenced to the platform coordinate system

${}^{P}\mathbf{C}_I$ couple vector exerted on the platform by the inertia of the platform; referenced to the platform coordinate system

${}^{P}\mathbf{C}_i$ couple vector exerted on the platform by leg i; referenced to the platform coordinate system

$\mathbf{D}, \mathbf{N}, \mathbf{S}$ vectors

$\mathbf{F}$ force vector

${}^{P}\mathbf{F}_E$ force vector exerted on the platform by the external world; referenced to the platform coordinate system

${}^{P}\mathbf{F}_G$ force vector exerted on the platform by the gravity field; referenced to the platform coordinate system

${}^{P}\mathbf{F}_I$ force vector exerted on the platform by the inertia of the platform; referenced to the platform coordinate system

$^{P}\mathbf{F}_i$ force vector exerted on the platform by leg i; referenced to the platform coordinate system

f magnitude of force vector; intensity of wrench

$\mathbf{f}$ vector made up of the magnitudes of the six forces exerted by the legs on the platform

$^{P}\mathbf{g}$ vector acceleration of the platform due to the gravity field; referenced to the platform coordinate system

h pitch of the *ISA*

h' pitch of wrench

$^{P}\mathbf{I}$ inertia tensor of the platform about the platform coordinate system

ISA instantaneous screw axis

$\mathbf{M}$ moment of vector about the origin normalized with respect to the magnitude of the vector

$\mathbf{M}'$ moment of vector about the origin

$\mathbf{M}_i$ normalized moment of vector $\mathbf{S}_i$ about the origin of the base coordinate system; moment vector of leg i in the base coordinate system; referenced to the base coordinate system

$^{P}\mathbf{M}_i$ normalized moment of vector $\mathbf{S}_i$ about the origin of the platform coordinate system; moment vector of leg i in the platform coordinate system; referenced to the platform coordinate system

$^{P}\mathbf{M}^*$ 3×6 matrix in which each column is the normalized moment vector of one of the legs; referenced to the platform coordinate system

m mass of the platform

$\mathbf{P}_i$ vector description of the end of leg i in the platform; referenced to the base coordinate system

$^{P}\mathbf{P}_i$ vector description of the end of leg i in the platform; referenced to the platform coordinate system

$\dot{\mathbf{P}}_i$ velocity of the point at the platform end of leg i; referenced to the base coordinate system

$\mathbf{Q}$ point

$\mathbf{R}$ rotation matrix that describes the orientation of the platform relative to the base; also used to transform free and line vectors from the base coordinate system to the platform coordinate system; referenced to the base coordinate system

$^{P}\mathbf{R}$ rotation matrix that describes the orientation of the base relative to the platform; also used to transform free and line vectors from the platform coordinate system to the base coordinate system; referenced to the platform coordinate system

R_x, R_y, R_z description of the orientation of the platform as rotations about the platform x-axis, y-axis, and z-axis

r_B distance from the origin of the base coordinate system to points B_i, $i = 1 \ldots 6$

r_P distance from the origin of the platform coordinate system to points P_i, $i = 1 \ldots 6$

$\mathbf{S}_i$ vector from the end of leg i in the base to the end of leg i in the platform; referenced to the base coordinate system

$^{P}\mathbf{S}_i$ vector from the end of leg i in the base to the end of leg i in the platform; referenced to the platform coordinate system

$\dot{\mathbf{S}}_i$ vector velocity of leg i; referenced to the base coordinate system

$^{P}\dot{\mathbf{S}}_i$ vector velocity of leg i; referenced to the platform coordinate system

$\mathbf{s}$ unit vector in the direction of vector $\mathbf{S}$

$\mathbf{s}_i$ unit vector along leg i in the direction from the base toward the platform; referenced to the base coordinate system

$^{P}\mathbf{s}_i$ unit vector along leg i in the direction from the base toward the platform; referenced to the platform coordinate system

$\dot{\mathbf{s}}_i$ component of the vector velocity of leg i that is perpendicular to the direction of the leg; vector rate of change of direction of leg i; referenced to the base coordinate system

$^{P}\dot{\mathbf{s}}_i$ component of the vector velocity of leg i that is perpendicular to the direction of the leg; vector rate of change of direction of leg i; referenced to the platform coordinate system

$^{p}\mathbf{s}^*$ 3×6 matrix in which each column is the unit vector of one of the legs; referenced to the platform coordinate system

σ_i magnitude of vector $\mathbf{S}_i$; length of leg i; actuator coordinate

$\dot{\sigma}_i$ magnitude of the component of the vector velocity of leg i that is parallel to the direction of the leg; rate of change of length of leg i; actuator velocity

$\mathbf{T}$ translation vector that describes the position of the platform relative to the base; the vector from the origin of the base coordinate system to the origin of the platform coordinate system; referenced to the base coordinate system

$^{p}\mathbf{T}$ translation vector that describes the position of the base relative to the platform; the vector from the origin of the platform coordinate system to the origin of the base coordinate system; referenced to the platform coordinate system

$\mathbf{U}$ Plücker coordinates of a line normalized with respect to the magnitude of the vector formed by the first three components of the Plücker coordinates; made up of the three components of the unit vector along the line and the three components of the moment of the unit vector about the origin

$\mathbf{U}'$ Plücker coordinates of a line; made up of the three components of a vector along the line and the three components of the moment of that vector about the origin

$\mathbf{U}_i$ normalized Plücker coordinates of leg i; referenced to the base coordinate system

$^{p}\mathbf{U}_i$ normalized Plücker coordinates of leg i; referenced to the platform coordinate system

$^{p}\mathbf{U}^*$ 6×6 matrix in which each column is the normalized Plücker coordinate vector of one of the legs; referenced to the platform coordinate system

$\mathbf{V}$ velocity of a body along the *ISA*

$\mathbf{V}'$ velocity of an arbitrary point in a body

$\boldsymbol{\omega}$ angular velocity vector of a body about the *ISA*

$^{p}\boldsymbol{\omega}$ angular velocity vector of the platform; referenced to the platform coordinate system

$^{p}\dot{\boldsymbol{\omega}}$ rate of change of angular velocity vector of the platform; angular acceleration vector of the platform; referenced to the platform coordinate system

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