Pergamon

*Mech. Mach. Theory* Vol. 29, No. 4, pp. 547–557, 1994

0094-114X/94 $6.00 + 0.00

# DISPLACEMENT ANALYSIS OF THE 6–6 STEWART PLATFORM MECHANISMS

FUAN WEN and CHONGGAO LIANG

Department of Mechanical Engineering, Beijing University of Posts and Telecommunications, 100088 Beijing, P.R.C.

*(Received 24 August 1992; in revised form 15 March 1993; received for publication 3 June 1993)*

**Abstract**—A method for the closed-form forward displacement analysis (FDA) of the Stewart platform mechanism (SPM) is presented in this paper. Meanwhile, the desired FDA in closed form of the 6–6/6–5/6–4 SPM with coplanar ball-joints has been performed. The analysis leads to a 20/20/16th order polynomial equation in one unknown, from which with other equations 40/40/32 different locations of the platform can be derived. This theory has been programmed in personal computer, all solutions can be gained in 4 s with seven digits of precision. The symbolic computation is carried out by computer algebra system REDUCE.

## 1. INTRODUCTION

Over the past few years, the parallel mechanisms (PM) have received a great deal of attention from many researchers. Compared to a serial mechanism, a PM possesses a stiff mechanical structure and consequent high natural frequencies, low sensitivity to external load variation on position accuracy, and high position accuracy of the output controlled link (platform). In situations where the needs for accuracy and sturdiness dominate the requirement of a large workspace, PM present themselves as viable alternatives to their serial counterparts.

Perhaps, a most well-known PM is the SPM which was introduced as an aircraft simulator by Stewart in 1965 [1]. Since then, Hunt, Mohamed, Duffy, Fichter, Sugimoto, Rees-Jones and Kerr all suggest use of SPM, with various applications ranging from manipulators to force/torque sensors.

Basically, a SPM consists of two platforms and 3–6 kinematic chains (legs) acting in parallel. One of the platforms is a movable link defined as the "top platform". It also can be named as "end effector" when the SPM is regarded as a parallel manipulator. It has six degrees of freedom relative to another platform named as "base" which also connected by the legs through joints when the input (a set of actuator displacements) alters freely. The SPM becomes a structure when the input is given.

A generalization of the SPM with discrete coplanar ball-joints, i.e. 6–6 SPM is illustrated in Fig. 1.

One of the reasons that PM's are not widely used in industry is that the FDA in closed form has not been solved. The FDA of PM is to seek the locations (position and orientation) of the top platform when the input is given. It is very complicated. It involves highly nonlinear equations which provide many solutions. It could be solved by iterative numerical methods that do not, however, guarantee finding all possible solutions. The number of solutions provides the upper limit for the number of possible real configurations of the top platform relative to the base for a specified set of actuator displacements. A closed-form FDA will provide more information about the geometry and kinematic behavior of a PM. This information is also extremely useful in practice for the control of PM and for the force control of force/torque sensors.

In terms of the geometry of top platform and base, the SPM can be classified into many cases. A class of SPM for which the both ends of the six legs where they joint the top platform and base are coplanar has been strongly investigated in recent years. For that class of SPM, the centers of the ball-joints generally form two hexagons on the top platform and base. When a ball-joint is concentrated on the other, the number of joining points of the six legs in the top and base platforms is respectively reduced. If the number in top platform is $n$, that in base is $N$, then the SPM is named as type $N$–$n$ SPM. If the number of the sides of the polygon which is formed by the centers of ball-joints on top platform is $m$, that in base is $M$, then the SPM is named as shape $M$–$m$ SPM. The fewer the number is, the easier to FDA. But, concentric ball-joints may cause design problems. To avoid the use of concentric ball-joints, the ball-joints of legs can be separated to become a higher type but keeping the same shape.

According to the arrangement of legs, the SPM also can be classified into another case. Some SPM can be analyzed by the same kinematic model, regardless of their leg arrangement.

In recent years much of the research in the literature has devoted extensive effort to the FDA. Griffis and Duffy solved the type 3–3 SPM in 1989 [2]. Since then, Rong and Liang (type 6–3) [3, 14], Liao and Liang (3-TPS) [5], Parenti-Castelli and Innocenti (RRR–3S, PPP–3S, PRR–3S, PPR–3S) [6, 7], Chang-de Zhang and Shin-Min Song (type 3–3) [8], Wei Lin *et al.* (type 4–4) [9] all presented closed form solutions. Huang Yu-Zhen and Wu Wen-da gave FDA of 6–6 SPM with special arrangement of joint centers [10]. Recently, Yin and Liang have gained the FDA of type 6–4 shape 3–3 SPM [11]. Xiao and Liang have gained the FDA of general type 6–4 [12]. Lee *et al.* have done that of the type 6/5/4–4 [4]. The FDA becomes more difficult with the increasing number of joining points of the six legs in the top and base platforms and the generalization of arrangement of joint centers. It can hardly be solved manually.

The aim of this paper is to present a novel method for the FDA in closed form. The desired FDA in closed form of SPM (type 6–6 shape 6–6) has been performed. We will present the procedure of analysis only, without the details due to space limitation.

## 2. STRUCTURAL PARAMETERS OF SPM

### *2.1. The coordinates of a spatial point can be described in terms of that of the other three points if they are all fixed in the same plane*

Referring to Fig. 2, $P_k$ is a point in the plane formed by points $P_1$, $P_2$ and $P_3$. The line $P_1P_k$ intersects with line $P_2P_3$ at $P_{k'}$. Let $P_2P_{k'}/P_{k'}P_3 = h_1$, $P_1P_k/P_kP_{k'} = h_2$, the coordinates of $P_k$ in $X_j$-axis is $X_{kj}$, then

$$X_{k'j} = (X_{2j} + h_1X_{3j})/(1 + h_1),$$

$$X_{kj} = (X_{1j} + h_2X_{k'j})/(1 + h_2),$$

$$X_{kj} = 1/(1 + h_2)X_{1j} + h_2/[(1 + h_1)(1 + h_2)]X_{2j} + h_1h_2/[(1 + h_1)(1 + h_2)]X_{3j},$$

$$X_{kj} = V_{k1}X_{1j} + V_{k2}X_{2j} + V_{k3}X_{3j}, \tag{1}$$

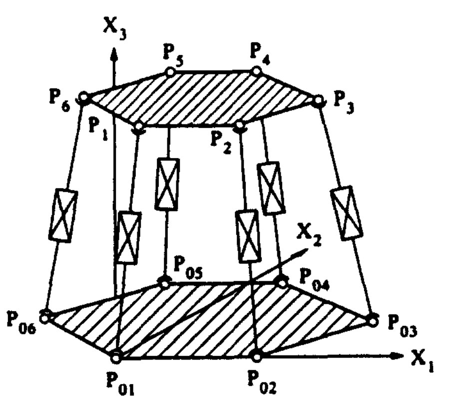


Fig. 1. The 6–6 SPM.

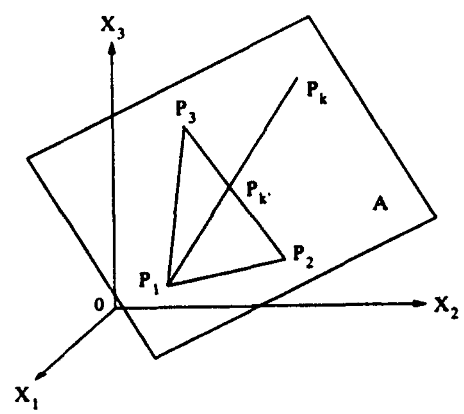


Fig. 2

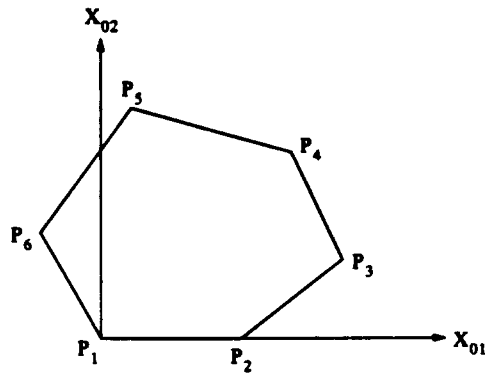


Fig. 3. The top platform of SPM.

where

$$V_{k1} = 1/(1+h_2),$$
$$V_{k2} = h_2/[(1+h_1)(1+h_2)],$$
$$V_{k3} = h_1 h_2/[(1+h_1)(1+h_2)],$$
$$V_{k1} + V_{k2} + V_{k3} = 1. \qquad (2)$$

$V_{k1}$, $V_{k2}$, $V_{k3}$ are only defined by the position of $P_k$ relative to that of points $P_1$, $P_2$, $P_3$.

### 2.2. Top platform

Referring to Fig. 3, we set up the movable coordinate system $X_{01}$–$X_{02}$ which is fixed at the top platform. Choose the joint center $P_1$ as the origin of the system, the $X_{01}$-axis passing through the joint center $P_2$, the $X_{02}$-axis in the top platform plane and perpendicular to the $X_{01}$-axis.

The coordinates of the joint center $P_i$ in top platform relative to the system $X_{01}$–$X_{02}$ are $X_{i0j}$ ($i = 1, 2, 3, 4, 5, 6$; $j = 01, 02$). They are the structural parameters of the SPM.

According to the definition of the system $X_{01}$–$X_{02}$, it is easy to know that

$$X_{101} = X_{102} = X_{202} = 0.$$

We can always choose a joint center above the $X_{01}$-axis as the $P_3$. Let the lengths of $P_iP_j$ be $D_{ij}$, so that

$$D_{12}^2 = (X_{101} - X_{201})^2 + (X_{102} - X_{202})^2,$$
$$D_{13}^2 = (X_{101} - X_{301})^2 + (X_{102} - X_{302})^2, \qquad (3)$$
$$D_{23}^2 = (X_{301} - X_{201})^2 + (X_{302} - X_{202})^2,$$

Replace $P_k$ by $P_4$, $P_5$, $P_6$ and $X_j$-axis by $X_{01}$-axis, $X_{02}$-axis in equation (1), then

$$X_{k01} = V_{k1}X_{101} + V_{k2}X_{201} + V_{k3}X_{301},$$
$$X_{k02} = V_{k1}X_{102} + V_{k2}X_{202} + V_{k3}X_{302}.$$

Combined with equation (2), we get

$$V_{k1} = (X_{201}X_{302} - X_{201}X_{k02} + X_{k02}X_{301} - X_{302}X_{k01})/(X_{201}X_{302}),$$
$$V_{k2} = (X_{302}X_{k01} - X_{k02}X_{301})/(X_{201}X_{302}),$$
$$V_{k3} = X_{k02}/X_{302} \quad (k = 4, 5, 6). \qquad (4)$$

### 2.3. Base platform

Now we set up the other coordinate system. Referring to Fig. 1, the three-dimensional coordinate system $X_1$–$X_2$–$X_3$ is fixed at the base. Choose the joint center $P_{01}$ as the origin of the system, the

$X_1$-axis passing through the joint center $P_{02}$, the $X_2$-axis in the base platform plane and perpendicular to the $X_1$-axis, the $X_3$-axis perpendicular to the base plane.

The absolute coordinates of the joint center $P_{0i}$ in base platform relative to the sytem $X_1$–$X_2$–$X_3$ are $X_{0ij}$ $(i = 1, 2, 3, 4, 5, 6;\ j = 1, 2, 3)$. They are also structural parameters of SPM.

It is easy to know that

$$X_{011} = X_{012} = X_{013} = X_{022} = X_{023} = X_{033} = 0.$$

Since the joint centers in base are coplanar so that

$$X_{043} = X_{053} = X_{063} = 0.$$

Similar to the top platform, the lengths of $P_{0i}P_{0j}$ are $B_{ij}$, so that

$$\begin{aligned} B_{12}^2 &= (X_{011} - X_{021})^2 + (X_{012} - X_{022})^2, \\ B_{13}^2 &= (X_{011} - X_{031})^2 + (X_{012} - X_{032})^2, \\ B_{23}^2 &= (X_{031} - X_{021})^2 + (X_{032} - X_{022})^2. \end{aligned} \tag{5}$$

Replace $P_{0k}$ by $P_{04}$, $P_{05}$, $P_{06}$ and $X_j$-axis by $X_1$-axis, $X_2$-axis in equation (1), then

$$\begin{aligned} X_{0k1} &= V_{0k1}X_{011} + V_{0k2}X_{021} + V_{0k3}X_{031}, \\ X_{0k2} &= V_{0k1}X_{012} + V_{0k2}X_{022} + V_{0k3}X_{032}. \end{aligned}$$

Combined with equation (2), we get

$$\begin{aligned} V_{0k1} &= (X_{021}X_{032} - X_{021}X_{0k2} + X_{0k2}X_{031} - X_{032}X_{0k1})/(X_{021}X_{032}), \\ V_{0k2} &= (X_{032}X_{0k1} - X_{0k2}X_{031})/(X_{021}X_{032}), \\ V_{0k3} &= X_{0k2}/X_{032} \quad (k = 4, 5, 6). \end{aligned} \tag{6}$$

## 3. INPUT AND UNKNOWN VARIABLES OF SPM

### 3.1. *The lengths of the six legs*

$$L_i = P_iP_{0i} \quad (i = 1, 2, 3, 4, 5, 6),$$

are input variables for the FDA, which are prescribed.

### 3.2. *The describing variables of the top platform*

It is very important to choose the appropriate variables for the description of the locations (position and orientation) of the top platform. The top platform has six degrees of freedom. At least six variables are needed to describe it. The previous paper on FDA all introduce the angle or direction cosines. We describe the locations with the absolute coordinates of some joint centers. Thus the procedure of the FDA may be simplified for there are no trigonometric functions in the equations.

The absolute coordinates (relative to the system $X_1$–$X_2$–$X_3$) of the joint center $P_i$ in top platform are $x_{ij}$ $(i = 1, 2, 3, 4, 5, 6;\ j = 1, 2, 3)$, which are unknown variables in FDA in this paper.

## 4. THE FDA IN CLOSED FORM OF THE 6–6 SPM

### 4.1. *Fundamental equations*

The distances between point $P_1$, $P_2$, $P_3$ are constant, i.e.

$$\begin{aligned} (x_{11} - x_{21})^2 + (x_{12} - x_{22})^2 + (x_{13} - x_{23})^2 - D_{12}^2 &= 0, \\ (x_{11} - x_{31})^2 + (x_{12} - x_{32})^2 + (x_{13} - x_{33})^2 - D_{13}^2 &= 0, \\ (x_{21} - x_{31})^2 + (x_{22} - x_{32})^2 + (x_{23} - x_{33})^2 - D_{23}^2 &= 0. \end{aligned} \tag{7}$$

The absolute coordinates of the joint center $P_4$, $P_5$, $P_6$ are linear with the absolute coordinates of the joint center $P_1$, $P_2$, $P_3$.

$$x_{kj} = V_{k1}x_{1j} + V_{k2}x_{2j} + V_{k3}x_{3j} \quad (k = 4, 5, 6;\ j = 1, 2, 3). \tag{8}$$

Since the coordinates of all joint centers $P_i$ and $P_{0i}$ are introduced as indicated above, we can write the equations of the lengths of legs in terms of the absolute coordinates of joint centers as follows:

$$\left.\begin{aligned}(x_{11})^2+(x_{12})^2+(x_{13})^2-L_1^2&=0\\(x_{21}-X_{021})^2+(x_{22})^2+(x_{23})^2-L_2^2&=0\\(x_{31}-X_{031})^2+(x_{32}-X_{032})^2+(x_{33})^2-L_3^2&=0\end{aligned}\right] \quad (9)$$

$$\left.\begin{aligned}(x_{41}-X_{041})^2+(x_{42}-X_{042})^2+(x_{43})^2-L_4^2&=0\\(x_{51}-X_{051})^2+(x_{52}-X_{052})^2+(x_{53})^2-L_5^2&=0\\(x_{61}-X_{061})^2+(x_{62}-X_{062})^2+(x_{63})^2-L_6^2&=0\end{aligned}\right] \quad (10)$$

Equations (7)–(10) are considered as the fundamental equations. There are 18 unknown variables $x_{ij}$ in these 18 equations, the $L_i$ are given as the input variables and the remainders are the structural parameters of the SPM.

*4.2. Reducing the fundamental equations to a polynomial in single variable*

Substituting equations (8) into equations (10) for $x_{kj}$, and using equations (9), then equations (7) and (10) are transformed into the following six equations:

$$\begin{aligned}x_{11}x_{21}+x_{12}x_{22}+x_{13}x_{23}-x_{21}X_{021}+A_{34}&=0,\\x_{11}x_{31}+x_{12}x_{32}+x_{13}x_{33}-x_{31}X_{031}-x_{32}X_{032}+A_{35}&=0\\x_{31}x_{21}+x_{32}x_{22}+x_{33}x_{23}-x_{21}X_{021}-x_{31}X_{031}-x_{32}X_{032}+A_{36}&=0,\end{aligned} \quad (11)$$

$$\begin{aligned}&V_{41}X_{041}x_{11}+V_{41}X_{042}x_{12}+V_{42}(X_{041}-X_{021})x_{21}+V_{42}X_{042}x_{22}+V_{43}(X_{041}-X_{031})x_{31}\\&\quad+V_{43}(X_{042}-X_{032})x_{32}+A_{37}=0,\\&V_{51}X_{051}x_{11}+V_{51}X_{052}x_{12}+V_{52}(X_{051}-X_{021})x_{21}+V_{52}X_{052}x_{22}+V_{53}(X_{051}-X_{031})x_{31}\\&\quad+V_{53}(X_{052}-X_{032})x_{32}+A_{38}=0,\\&V_{61}X_{061}x_{11}+V_{61}X_{062}x_{12}+V_{62}(X_{061}-X_{021})x_{21}+V_{62}X_{062}x_{22}+V_{63}(X_{061}-X_{031})x_{31}\\&\quad+V_{63}(X_{062}-X_{032})x_{32}+A_{39}=0,\end{aligned} \quad (12)$$

where $A_{3i}$s are rational functions of the $L_i$s and the structural parameters.

By choosing $P_{01}$, $P_{02}$, $P_{03}$ properly, the coefficient matrix of linear equations (12) in the variables $x_{11}$, $x_{21}$, $x_{31}$ is nonsingular. Solving these equations with respect to these three variables, we get

$$\begin{aligned}x_{11}&=x_{12}A_{441}+x_{22}A_{442}+x_{32}A_{443}+A_{444},\\x_{21}&=x_{12}A_{451}+x_{22}A_{452}+x_{32}A_{453}+A_{454},\\x_{31}&=x_{12}A_{461}+x_{22}A_{462}+x_{32}A_{463}+A_{464},\end{aligned} \quad (13)$$

where the properties of $A_{4ij}$s are similar to $A_{3i}$s.

Separating the variables $x_{13}$, $x_{23}$, $x_{33}$ from others in equations (9) and (11), we obtain

$$\begin{aligned}(x_{13})^2&=-(x_{11})^2-(x_{12})^2+L_1^2,\\(x_{23})^2&=-(x_{21}-X_{021})^2-(x_{22})^2+L_2^2,\\(x_{33})^2&=-(x_{31}-X_{031})^2-(x_{32}-X_{032})^2+L_3^2,\end{aligned} \quad (14)$$

$$\begin{aligned}x_{13}x_{23}&=-x_{11}x_{21}-x_{12}x_{22}+x_{21}X_{021}-A_{37},\\x_{13}x_{33}&=-x_{11}x_{31}-x_{12}x_{32}+x_{31}X_{031}+x_{32}X_{032}-A_{38},\\x_{33}x_{23}&=-x_{31}x_{21}-x_{32}x_{22}+x_{21}X_{021}+x_{31}X_{031}+x_{32}X_{032}-A_{39}.\end{aligned} \quad (15)$$

Seeing that the left-hand sides of equations (14) and (15) are related each other, we get the following six equations:

$$\begin{aligned}(14.1)\cdot(15.3)-(15.1)\cdot(15.2)&=0,\\ (14.2)\cdot(15.2)-(15.1)\cdot(15.3)&=0,\\ (14.3)\cdot(15.1)-(15.2)\cdot(15.3)&=0,\\ (14.1)\cdot(14.2)-(15.1)^2&=0,\\ (14.1)\cdot(14.3)-(15.2)^2&=0,\\ (14.2)\cdot(14.3)-(15.3)^2&=0.\end{aligned}\tag{16}$$

Substituting equations (13) into equations (14) and (15) for $x_{j1}$, then substituting right-hand sides of equations (14) and (15) into equations (16). We get six equations in terms of $x_{12}$, $x_{22}$, $x_{32}$. The total degree of every equation with respect to $x_{12}$, $x_{22}$ and $x_{32}$ is 4.

$$\sum_{j,k,m=0}^{4} A_{ijkm}x_{12}^{j}x_{22}^{k}x_{32}^{m}=0 \quad (i=1,2,3,4,5,6), \tag{17A}$$

where $j+k+m\leqslant 4$. Similarly to $A_{3i}$s, the $A_{ijkm}$s are rational functions of the knowns. Separating the terms $x_{12}^{j}x_{22}^{4-j}$ from others in equations (17A), we obtain

$$\begin{bmatrix} A_{1400} & A_{1310} & A_{1220} & A_{1130} & A_{1040}\\ A_{2400} & A_{2310} & A_{2220} & A_{2130} & A_{2040}\\ A_{3400} & A_{3310} & A_{3220} & A_{3130} & A_{3040}\\ A_{4400} & A_{4310} & A_{4220} & A_{4130} & A_{4040}\\ A_{5400} & A_{5310} & A_{5220} & A_{5130} & A_{5040}\\ A_{6400} & A_{6310} & A_{6220} & A_{6130} & A_{6040} \end{bmatrix} \begin{bmatrix} x_{12}^4\\ x_{12}^3x_{22}\\ x_{12}^2x_{22}^2\\ x_{12}x_{22}^3\\ x_{22}^4 \end{bmatrix} + \begin{bmatrix} q_{71}\\ q_{72}\\ q_{73}\\ q_{74}\\ q_{75}\\ q_{76} \end{bmatrix} = 0. \tag{17B}$$

where $q_{7i}$s are also polynomials in $x_{12}$, $x_{22}$, $x_{32}$. Obviously, the total degree of every polynomial of $q_{7i}$s with respect to $x_{12}$ and $x_{22}$ is only 3.

It is easy to calculate that the rank of the $6\times 5$ coefficient matrix in equation (17B) is equal to five in most cases such as type 6–6, 6–5 and 6–4. But for type 4–4, 6–3, 3–3 and Huang's shape of type 6–6, the rank of the matrix is less than five, that means, this method is disabled in these cases.

The combinations are arranged so that the coefficient matrix becomes identity

$$\begin{aligned}x_{12}^4+q_{81}&=0,\\ x_{12}^3x_{22}+q_{82}&=0,\\ x_{12}^2x_{22}^2+q_{83}&=0,\\ x_{12}x_{22}^3+q_{84}&=0,\\ x_{22}^4+q_{85}&=0,\\ q_{86}&=0,\end{aligned}\tag{18}$$

where the properties of $q_{8i}$ are similar to $q_{7i}$.

Multiplying equations (18) by $x_{12}$ and $x_{22}$, we get

$$\begin{aligned}x_{12}^5+x_{12}q_{81}&=0,\\ x_{12}^4x_{22}+x_{12}q_{82}&=0,\\ x_{12}^3x_{22}^2+x_{12}q_{83}&=0,\\ x_{12}^3x_{22}^3+x_{12}q_{84}&=0,\\ x_{12}x_{22}^4+x_{12}q_{85}&=0,\\ x_{12}q_{86}&=0,\end{aligned}$$

$$
\begin{aligned}
x_{12}^4 x_{22} + x_{22} q_{81} &= 0,\\
x_{12}^3 x_{22}^2 + x_{22} q_{82} &= 0,\\
x_{12}^2 x_{22}^3 + x_{22} q_{83} &= 0,\\
x_{12} x_{22}^4 + x_{22} q_{84} &= 0,\\
x_{22}^5 + x_{22} q_{85} &= 0,\\
x_{22} q_{86} &= 0,\\
x_{12}^2 q_{86} &= 0,\\
x_{12} x_{22} q_{86} &= 0,\\
x_{22}^2 q_{86} &= 0.
\end{aligned}
\tag{19}
$$

These 15 equations, combined with the six equations of equation (18), form the 21 resultant equations. They can be written in the form of matrix as follows

$$[B][X] = 0, \tag{20}$$

where $[B]$ is a $21 \times 21$ matrix in which the elements are polynomial in $x_{32}$ and $[X]$ is a column matrix in which there are 21 false variables,

$$[X] = [x_{12}^5, x_{12}^4 x_{22}, x_{12}^4, x_{12}^3 x_{22}^2, x_{12}^3 x_{22}, x_{12}^3, x_{12}^2 x_{22}^3, x_{12}^2 x_{22}^2, x_{12}^2 x_{22}, x_{12}^2, x_{12} x_{22}^4, x_{12} x_{22}^3, x_{12} x_{22}^2, x_{12} x_{22}, x_{12}, x_{22}^5, x_{22}^4, x_{22}^3, x_{22}^2, x_{22}, 1]^{\mathrm{T}}.$$

For the matrix $[X]$ is not always to be equal to zero, the determinant of the square matrix $[B]$ must be zero

$$\mathrm{Det}[B] = 0. \tag{21}$$

This is the unary polynomial in $x_{32}$ which is the input–output equation that we are trying to find.

Noticing that some elements in the Det$[B]$ are 1, the other elements in the same column are made zero by adding proper multiples of the row which involve the "1". Finally, 11 columns in the Det$[B]$ are diagonalizable and we get

$$(1)^{11}\,\mathrm{Det}[M] = \mathrm{Det}[B] = 0, \tag{22}$$

where the Det$[M]$ is the 10th degree determinant of matrix $[M]$ which corresponds to the equation

$$[M][XA] = 0, \tag{23}$$

where

$$[XA] = [x_{12}^3, x_{12}^2 x_{22}, x_{12}^2, x_{12} x_{22}^2, x_{12} x_{22}, x_{12}, x_{22}^3, x_{22}^2, x_{22}, 1]^{\mathrm{T}}$$

so that

$$\mathrm{Det}[M] = 0. \tag{24}$$

It is discovered that the highest degree term of the polynomial of every element in $[M]$ is linearly dependent. Elementary row operations are performed in $[M]$ so that the highest degree term of the polynomial in most elements in $[M]$ is made zero. Then the degree of the polynomial of every element is shown in following table

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |
| 1, | 1, | 2, | 1, | 2, | 3, | 1, | 2, | 3, | 4 |

From this table, it is obvious that the degree of the input–output equation in the form of Det[$M$] in $x_{32}$ is at most 20.

The Det[$M$] can be calculated by computer algebraic system such as "Mathematica". In most cases, it is a 20th degree polynomial. That also means the variable $x_{32}$ has 20 solutions. But in some special cases, say type 6–4, the degree of the polynomial is reduced to 16.

### *4.3. Compute the other variables*

Taking any nine from the 10 equations of equation (23) which now are considered to be linear equations in the variables $[x_{12}^3, x_{12}^2x_{22}, x_{12}^2, x_{12}x_{22}^2, x_{12}x_{22}, x_{12}, x_{22}^3, x_{22}^2, x_{22}]$, we can get the expressions for $x_{12}$ and $x_{22}$ as rational functions in terms of $x_{32}$, say by Cramer's rule. For one solution of $x_{32}$ there will be one solution of $x_{12}$ or $x_{22}$.

Substituting $x_{32}$, $x_{12}$, $x_{22}$ into equations (13), the solutions of $x_{11}$, $x_{21}$, $x_{31}$ can be gained. For one solution of $x_{32}$ there will be one solution of $x_{11}$, $x_{21}$ or $x_{31}$.

Using equations (14), we can get the solutions of $x_{13}$, $x_{23}$, $x_{33}$. For one solution of $x_{32}$, there will be two opposite sign solutions of each $x_{13}$, $x_{23}$ or $x_{33}$. But they can not be combined randomly to be the solutions of the fundamental equations. They must satisfy equations (15). So there are only two groups of solutions of $x_{13}$, $x_{23}$, $x_{33}$ which respond to one solution of $x_{32}$.

## 5. USING THE PROGRAM "SPM FDA SOLVER"

### *5.1. Calculating a type 6–6 shape 6–6 SPM*

Being an example to check our method, the coordinates of $P_j$ relative to the $X_{01}$–$X_{02}$ system are given as follows:

$$X201 = 4 \quad X301 = 3 \quad X302 = 3$$
$$X401 = 2 \quad X501 = 6 \quad X601 = 1$$
$$X402 = -1 \quad X502 = 2 \quad X602 = 2.$$

The coordinates of $P_{0j}$ relative to the $X_1$–$X_2$–$X_3$ system are given as

$$X021 = 6 \quad X031 = 2 \quad X032 = 4$$
$$X041 = 7/2 \quad X051 = 27/4 \quad X061 = -1/2$$
$$X042 = -2 \quad X052 = 3 \quad X062 = 2.$$

We call this description of parameters as in $X$ mode. Now translate it into so called $V$ mode.

| | | |
|---|---|---|
| $D12 = 4$ | $D13 = 4.24264068711929$ | $D23 = 3.16227766016838$ |
| $V41 = 0.583333333333333$ | $V51 = -0.666666666666667$ | $V61 = 0.583333333333333$ |
| $V42 = 0.75$ | $V52 = 1$ | $V62 = -0.25$ |
| $V43 = -0.333333333333333$ | $V53 = 0.666666666666667$ | $V63 = 0.666666666666667$ |
| $B12 = 6.$ | $B13 = 4.47213595499958$ | $B23 = 5.65685424949238$ |
| $W41 = 0.75$ | $W51 = -0.625$ | $W61 = 0.75$ |
| $W42 = 0.75$ | $W52 = 0.875$ | $W62 = -0.25$ |
| $W43 = -0.5$ | $W53 = 0.75$ | $W63 = 0.5$ |

The lengths of six legs are given as follows:

$$L1 = 10.0995049383621 \quad L2 = 10.0995049383621 \quad L3 = 10.198039027185$$
$$L4 = 10.2102889283311 \quad L5 = 10.0031245118713 \quad L6 = 10.356157588604.$$

Using the above values as the input, from our program "SPM FDA SOLVER" which is coded with the above theory, we get the unary polynomial in $x_{32}$.

$$p(x_{32}) = 1.122 * 10\hat{}40 + 1.433 * 10\hat{}40 * x - 1.971 * 10\hat{}40 * x\hat{}2 + 6.594 * 10\hat{}39 * x\hat{}3$$
$$- 2.162 * 10\hat{}38 * x\hat{}4 - 3.097 * 10\hat{}38 * x\hat{}5 + 6.356 * 10\hat{}37 * x\hat{}6$$
$$- 2.395 * 10\hat{}36 * x\hat{}7 - 6.571 * 10\hat{}35 * x\hat{}8 + 5.968 * 10\hat{}34 * x\hat{}9 + 1.505 * 10\hat{}34 *$$
$$\times x\hat{}10 - 5.034 * 10\hat{}33 * x\hat{}11 + 7.669 * 10\hat{}32 * x\hat{}12 - 7.659 * 10\hat{}31 * x\hat{}13$$
$$+ 5.478 * 10\hat{}30 * x\hat{}14$$
$$- 2.885 * 10\hat{}29 * x\hat{}15 + 1.119 * 10\hat{}28 * x\hat{}16 - 3.108 * 10\hat{}26 * x\hat{}17$$
$$+ 5.837 * 10\hat{}24 * x\hat{}18 - 5.255 * 10\hat{}22 * x\hat{}19 - 2.825 * 10\hat{}19 * x\hat{}20,$$

where the "*" stand for multiple, the "10^n" stand for "$10^n$".

The all solutions of FDA are

$$x11 = \{1704., -40.83, 3.106 - 22.07 * I, 3.106 + 22.07 * I, -4.567 - 6.015 * I,$$
$$- 4.567 + 6.015 * I, 6.427 - 4.584 * I, 6.427 + 4.584 * I, -0.995 - 2.94 * I,$$
$$-0.995 + 2.94 * I, 2.72 - 9.321 * I, 2.72 + 9.321 * I, -5.996, 14.18, 1.794$$
$$- 1.419 * I, 1.794 + 1.149 * I, 5.267, -0.7385 + 3.362 * I,$$
$$- 0.7385 - 3.362 * I, 1.\}$$

$$x21 = \{1444., -42.38, 4.559 - 16.82 * I, 4.559 + 16.82 * I, 0.2641 - 2.867 * I,$$
$$0.2641 + 2.867 * I, 15.37 + 3.159 * I, 15.37 - 3.159 * I, -4.008 + 0.2843 * I,$$
$$- 4.008 - 0.2843 * I, 1.789 - 7.445 * I, 1.789 + 7.445 * I, \quad -0.9003, 17.66,$$
$$6.65 - 2.285 * I, 6.65 + 2.285 * I, 8.626, 3.038 + 4.613 * I, 3.038 - 4.613 * I, 5.\}$$

$$x31 = \{1893., -44.77, -1.716 - 19.5 * I, -1.716 + 19.5 * I, 1.685 - 10.86 * I,$$
$$1.685 + 10.86 * I, 4.301 + 3.167 * I, 4.301 - 3.167 * I, 2.787 + 1.561 * I,$$
$$2.787 - 1.561 * I, -1.329 - 8.66 * I, -1.329 + 866 * I, -4.382, 13.42,$$
$$3.542 - 4.33 * I, 3.542 + 4.33 * I, 8.218, 4.007 + 2.663 * I, 4.007 - 2.663 * I, 4.\}$$

$$x12 = \{-1733., 38.87, -0.5599 + 22.31 * I, -0.5599 - 22.31 * I, 5.918 + 7.884 * I,$$
$$5.918 - 7.884 * I, -0.939 + 6.119 * I, -0.939 - 6.119 * I, -0.3143 + 3.264 * I,$$
$$- 0.3143 - 3.264 * I, -1.376 + 9.358 * I, -1.376 - 9.358 * I, 8.25, -10.77,$$
$$0.8702 + 1.506 * I, 0.8702 - 1.506 * I, -3.238, 2.188 - 2.457 * I,$$
$$2.188 + 2.457 * I, 1.\}$$

$$x22 = \{-1389., 34.18, 9.608 + 11.84 * I, 9.608 - 11.84 * I, 0.2087 + 11.52 * I,$$
$$0.2087 - 11.52 * I, 6.559 - 5.115 * I, 6.559 + 5.115 * I, -1.397 - 3.26 * I,$$
$$- 1.397 + 3.26 * I, 7.914 + 5.467 * I, 7.914 - 5.467 * I, 7.592, -2.049,$$
$$2.555 + 4.581 * I, 2.555 - 4.581 * I, -1.774, -0.4318 - 0.6377 * I,$$
$$-0.4318 + 0.6377 * I, 1.\}$$

$$x32 = \{-1968., 37.06, 5.554 + 23.79 * I, 5.554 - 23.79 * I, 6.939 + 13.98 * I,$$
$$6.939 - 13.98 * I, 11.3 + 8.166 * I, 11.3 - 8.166 * I, -5.277 + 3.118 * I,$$

$$-5.277-3.118*I, 1.701+9.929*I, 1.701-9.929*I, 12.69, -4.389,$$
$$5.713+2.538*I, 5.713-2.538*I, -0.4547, 3.556-0.3531*I,$$
$$3.556+0.3531*I, 4.\}$$
$$x13=\pm\{2430.*I, 55.46*I, 32.91+2.463*I, 32.91-2.463*I, 13.26-5.592*I,$$
$$13.26+5.592*I, 11.31+3.113*I, 11.31-3.113*I, 10.97-0.1732*I,$$
$$10.97+0.1732*I, 16.51+2.316*I, 16.51-2.316*I, 1.42*I, 14.67*I,$$
$$10.12+0.1221*I, 10.12-0.1221*I, 7.986, 10.7+0.7341*I,$$
$$10.7-0.7341*I, 10.\}$$
$$x23=\pm\{1999.*I, 58.37*I, 21.71-6.358*I, 21.71+6.358*I, 14.55-1.295*I,$$
$$14.55+1.295*I, 2.997+1.314*I, 2.997-1.314*I, 3.295-0.5189*I,$$
$$3.295+0.5189*I, 12.05-6.19*I, 12{,}05+6.19*I, -1.805*I, 6.17*I,$$
$$11.05-0.9247*I, 11.05+0.9247*I, 9.589, 10.78+1.242*I,$$
$$10.78-1.242*I, 10.\}$$
$$x33=\pm\{2732.*I, 56.36*I, 32.33-3.384*I, 32.33+3.384*I, 20.33-2.189*I,$$
$$20.33+2.189*I, 12.31-5.432*I, 12.31+5.432*I, 6.79+4.078*I,$$
$$6.79-4.078*I, 16.17-0.3713*I, 16.17+0.3713*I, 3.495*I, 9.836*I,$$
$$11.13+0.2092*I, 11.13-0.2092*I, 6.745, 10.36-0.5313*I,$$
$$10.36+0.5313*I, 10.\}$$

All that calculations can be gained in 4 s on a personal computer (CPU 80386). The precision will rise to 13 digits with the run time increasing. The results fully justify all the roots that are not extraneous.

### 5.2. *The particular cases of SPM*

When a ball-joint $P_i$ is on the line of the opposite side of $P_j$, the $V_{ij}$ equals zero. When $P_i$ coincides with $P_j$, the $V_{ij}$ equals 1 and the other two $V_{ik}$ $(k \neq j)$ equal 0. For example, if $P_4$ is on the line of the opposite side of $P_3$, $V_{43}$ becomes 0. The SPM becomes 6–5 shape. If $P_4$ coincides with point $P_1$, $V_{41}$ becomes 1. $V_{42}$, $V_{43}$ become 0, then the SPM becomes 6–5 type.

The 6–6 type, 6–5 type and 5–5 type all have 40 solutions.

The 6–4 type has 32 solutions.

## 6. CONCLUSIONS

This paper presents a promising method of FDA for the 6–6/6–5/6–4 SPM.

The desired FDA in closed form for most SPM with coplanar ball-joints has been performed.

The all solutions for 6–6/6–6 SPM are within 40 groups.

This method has been successfully performed in our program "SPM FDA SOLVER".

*Acknowledgement*—The financial support of the National Natural Science Foundation of China is gratefully acknowledged.

## REFERENCES

1. D. Stewart, *Proc. Inst. Mech. Eng.* London, Vol. 180, pp. 371–386 (1965).
2. M. Griffis and J. Duffy, *J. Robot. Systems* **6**, 703–720 (1989).
3. H. Rong and C. G. Liang, *Proc. 7th Mechanisms Conf. of China*, Hebei, Pt. 2, pp. 36–40 (1990).
4. Hong-You Lee, Wei Lin and J. Duffy, A method for forward displacement analysis of in-parallel platform mechanisms. In communication (1992).
5. Q. Z. Liao and C. G. Liang, *Proc. 7th Mechanisms Conf. of China*, Hebei, Pt. 1, pp. 257–260 (1990).
6. V. Parenti-Castelli and C. Innocenti, *21st Biennial ASME Mechanisms Conf.*, Vol. 2, pp. 111–116 (1990).
7. C. Innocenti and V. Parenti-Castelli, *Mech. Mach. Theory* **25**, 611–621 (1990).

8. Chang-de Zhang and Shin-Min Song, *21st Biennial ASME Mechanisms Conf.*, Vol. 2, pp. 271–278 (1990).
9. Wei Lin, J. Duffy and M. Griffis, *21st Biennial ASME Mechanisms Conf.*, Pt. 2, pp. 263–269 (1990).
10. Huang Yu-Zhen and Wu Wen-da, MM Research Preprints No. 6, pp. 94–101 (1991).
11. J. P. Yin and C. G. Liang, *8th Mechanisms Conf. of China*, Shanghai (1992).
12. L. Z. Xiao and C. G. Liang, *8th Mechanisms Conf. of China*, Shanghai (1992).
13. K. H. Hunt, *Trans. ASME J. Mech. Transm. Automn Des.* **105**, 705–712 (1983).
14. H. Rong and C. G. Liang, *Proc. 8th Wld Congr. Theory of Machines and Mechanisms*, Prague, Vol. 4, pp. 1237–1240 (1991).
15. C. G. Liang, H. Y. Lee and Q. Z. Liao, *Analysis of Spatial Linkages and Robot Mechanisms*. B.U.P.T. Publishing House, Beijing, China (1988).
16. K. H. Hunt and E. J. F. Primrose. *Mech. Mach. Theory* **28**, 31–42 (1993).

**Zusammenfassung**—Zur Formulierung der Polynomgleichung mit kleistmöglichen Grad in einer Unbekannten für die Vorwärtstransformation von parallelen Plattformmechanismen wird in der vorliegenden Arbeit eine Methode entwickelt. Die Methode, die auf nur einem einzigen Modell beruht, ermoglicht die analytishs Untersuchung aller parallelen Plattformmechanismen, deren obere Platform in vier, funf, oder sechs Punkten mit einer ebenen Basis durch sechs SPS kinematische Kette verbunden ist. Die Anzahl der Verbindungspunkte in der Basis kann funf oder sechs sein. Als numerische Beispiele werden einige Plattformmechanismen untersucht, für die bisher noch keine Polynomgleichungen mit kleinstmoglichen Grad ermittelt worden sind.